SEVERAL COMPLEX VARIABLES

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SEVERAL COMPLEX VARIABLES

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Preface

The object of this book is to give an account of some of the important elements of the theory of functions of several complex variables. The book presupposes no knowledge of analytic functions of several variables, and there is very little duplication of material presented in the books by S. Bergman, by H. Behnke and P. Thullen, and by W. F. Osgood.

For the most part, the reader requires only a general knowledge of analysis, of the form contained in a rigorous text on advanced calculus together with the elementary concepts associated with the Lebesgue integral. In particular only the most elementary concepts of analytic functions of one complex variable are used. In Chapters I and III, the concepts of groups and semi-groups are used and certain notions of general analysis such as Haar-measure enter in. In Chapter VI some knowledge of Fourier analysis will be helpful. The treatment of Chapter X is algebraic and requires general knowledge of ideal theory.

The reader who wishes a quick introduction to the theory of analytic functions of several variables may start with Chapter II. Here basic facts about analytic functions of real and complex variables are presented.

The work on transformations is divided into two parts. Chapter I is on groups of transformations by formal power series with no convergence or analyticity assumed. In Chapter III convergence requirements are added and the results of Chapter I are carried to analytic mappings. The remaining chapters are on the whole independent of Chapters I and III.

This book was planned and begun in 1940; much of the work on it took place during the period 1940–1942. The material of several sections was discussed in detail by us with Dr. D. C. May, Jr. during 1940–1941. This applies particularly to the material on coordinate spaces in Chapter III and to the results on removable singularities of analytic functions presented in Chapter VIII. These discussions were very stimulating and helpful to the authors. The material of Chapter X is taken without essential change from the appendix, pages 257–269, of the Princeton University notes, "Functions of Several Complex Variables," lectures by S. Bochner, Princeton, 1936. This

part of the notes was written up in 1936 by Mr. S. Wylie and is based upon a paper by W. Rückert.

The second named author wishes to express his thanks to the Massachusetts Institute of Technology and to Princeton University for having made it possible for him to spend the year 1940–1941 at Princeton. It was during this year that the bulk of the work on the book was done.

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SEVERAL COMPLEX VARIABLES

CHAPTER I

Groups of Transformations by Formal Power-series

§1. Calculus of Formal Power-series

If $x_1, \cdots, x_k$ are given variables, then a formal power-series is an expression of the form

$$(1) \qquad f(x_1, \cdots, x_k) \equiv \Sigma_{n_1, \cdots, n_k=0}^{\infty} a_{n_1 \cdots n_k} x_1^{n_1} \cdots x_k^{n_k}$$

with any complex coefficients $\{a\}$. The series is said to *vanish* if all of its coefficients are zero. If

$$(2) \qquad g(x_1, \cdots, x_k) = \Sigma_n a'_{n_1 \cdots n_k} x_1^{n_1} \cdots x_k^{n_k}$$

is another series then the linear combination (α, β are constants)

$$(3) \qquad \alpha f + \beta g = \Sigma_n a''_{n_1 \cdots n_k} x_1^{n_1} \cdots x_k^{n_k}$$

and the product

$$(4) \qquad fg = \Sigma_n a'''_{n_1 \cdots n_k} x_1^{n_1} \cdots x_k^{n_k}$$

are defined by

$$a''_{n_1 \cdots n_k} = \alpha a_{n_1 \cdots n_k} + \beta a'_{n_1 \cdots n_k}$$

and

$$a'''_{n_1 \cdots n_k} = \Sigma_{\mu_1+\nu_1=n_1} \cdots \Sigma_{\mu_k+\nu_k=n_k} a_{\mu_1 \cdots \mu_k} a'_{\nu_1 \cdots \nu_k}$$

respectively. It can be easily verified that the multiplication (4) is associative and commutative. Thus the totality of all series (1) is a commutative ring.

The sum $n = n_1 + \cdots + n_k$ in any individual monomial in (1) will be called its *order*. Likewise this sum n will be called the order of the coefficient of that monomial. Denoting by $A_n(x_1, \cdots, x_k)$ the (finite) polynomial consisting of all terms of order n, we may also write (1) in the form $\Sigma_{n=0}^{\infty} A_n(x)$. By writing it in the form $\Sigma_{n=p}^{\infty} A_n$, where p is some integer > 0, we will indicate that all terms of order $\leq p - 1$ happen to vanish. If two series

$$\Sigma_{n=p}^{\infty} A_n, \qquad \Sigma_{n=q}^{\infty} A'_n$$

are given then their product starts formally with the polynomial

$A_p A'_q$, which is of order $p + q$, whereas all other terms are of higher order. Now, by ordinary algebra, if the product $A_p A'_q$ vanishes (identically) then at least one factor does. Hence we can easily deduce that the product fg cannot vanish unless one of the factors vanishes. In other words *our ring has no null-divisors.*

Now let

$$(5) \qquad y_\lambda = f_\lambda(x_1, \cdots, x_k), \qquad \lambda = 1, \cdots, l$$

be a finite number of series, and let

$$(6) \qquad g(y_1, \cdots, y_l) = \Sigma_{\nu=0}^{\infty} B_\nu(y)$$

be a series in $y_1, \cdots, y_l$, again with complex coefficients. If (6) is a polynomial in $y_1, \cdots, y_l$, then we can immediately form the series

$$(7) \qquad g[f_1(x), \cdots, f_l(x)]$$

and on denoting the coefficients of the series (5), (6), (7) by

$$a^\lambda_{p_1 \cdots p_k}, \; b_{q_1 \cdots q_l}, \; a_{r_1 \cdots r_k}$$

respectively we obtain

$$(8) \qquad a_{n_1 \cdots n_k} = \alpha_{n_1 \cdots n_k}(a^\lambda_p, b_q)$$

where each expression $\alpha(a^\lambda_p, b_q)$ is a polynomial in the quantities a^λ_p with coefficients b_q. But if (6) is an infinite series the corresponding expression (8) is in general also an infinite symbol and this is not suitable for the analysis we have in mind. However, if the series (5) have no constant terms, that is, if

$$(9) \qquad f_\lambda(x) = \Sigma_{n=1}^{\infty} A^\lambda_n(x)$$

then (8) is again a finite polynomial. In fact if $n_1 + \cdots + n_k = n$ then a quantity a^λ_p or a quantity b_q cannot actually occur in (8) unless its order is $\leq n$. We now agree that in the course of the present chapter *only power-series with no constant term will be considered. The ring of those series will be denoted by R, $R\{x\}$, $R\{y\}$, $R\{x_1, \cdots, x_k\}$, $R\{y_1, \cdots, y_l\}$, etc.*

Thus we can state that if $f_\lambda(x) \in R\{x\}$ and $g(y) \in R\{y\}$, then (7) belongs to $R\{x\}$. Now take several series

$$(10) \qquad z_\mu = g_\mu(y_1, \cdots, y_l), \qquad \mu = 1, \cdots, m$$

belonging to $R\{y\}$ and an element

$$t = h(z_1, \cdots, z_m) \tag{11}$$

from $R\{z\}$. We can obtain an element in $R\{x\}$ in two different ways. We first form the series

$$\varphi_\mu(x) = g_\mu[f_1(x), \cdots, f_l(x)]$$

and then

$$h[\varphi_1(x), \cdots, \varphi_m(x)] \tag{12}$$

or, we first form the series

$$\psi(y) = h[g_1(y), \cdots, g_m(y)]$$

and then

$$\psi[f_1(x), \cdots, f_l(x)] \tag{13}$$

If the series (5), (10), (11) are all finite, that is if in each only a finite number of coefficients are different from zero, then (12) and (13) are both polynomials. In this case they obviously have the same numerical values for all numerical values of the "variables" $x_1, \cdots, x_k$, and hence they are equal in all their coefficients. For the general case, on denoting the coefficients of (5), (10), (11) by

$$a_p^\lambda, \qquad b_q^\mu, \qquad c_r \tag{14}$$

respectively, and those of (12) and (13) by ϵ_n and δ_n respectively we have on the one hand

$$\epsilon_{n_1 \ldots n_k} \equiv \epsilon_{n_1 \ldots n_k}(a_p^\lambda, b_q^\mu, c_r) \tag{15}$$
$$\delta_{n_1 \ldots n_k} \equiv \delta_{n_1 \ldots n_k}(a_p^\lambda, b_q^\mu, c_r) \tag{16}$$

where the expressions on the right are *polynomials* in the quantities (14). On the other hand we have seen that for any multi-index $(n_1, \cdots, n_k)$ these polynomials have equal values for all possible combinations of numerical values for the quantities (14) as long as only a finite number of the latter values are different from zero. Hence for each multi-index the polynomials (15) and (16) are equal for all possible numerical values of their arguments and this proves the equality of (12) and (13) in the general case.

Now assume that a finite number of elements

$$t_\nu = h_\nu(z_1, \cdots, z_m), \qquad \nu = 1, \cdots, n \tag{17}$$

of $R\{z\}$ are given, and interpret relations (5) as a formal analytic transformation T from the space of the x-coordinates into the space

of the y-coordinates which carries the origin into the origin, and similarly interpret relations (10) and (17) as transformations U and V from the y-space into the z-space and from the z-space into the t-space. Then the statement just proved amounts to verifying the associative law

$$V(UT) = (VU)T \tag{18}$$

for combining those transformations successively. In particular if $n = m = k = l$ we may let the different spaces coincide coordinate by coordinate in which case each transformation is an "inner transformation" or an "inner map" of the space into itself. Any two transformations T and U, if carried out successively, give a new transformation UT, and UT may be different from TU; however, the associative law (18) always holds. Thus: *the family of all inner transformations forms a semigroup.* The family contains a unit element, namely the identity:

$$y_j = x_j, \qquad j = 1, \cdots, k \tag{19}$$

Our next step will be to find the inverse of an inner transformation. This will amount to solving a system of equations (5) when $l = k$. We will handle the more general situation of implicit functions, and we will start with the special case of one "unknown." Thus we take one variable x and several variables $y_1, \cdots, y_k$, and we consider an element

$$F(x; y_1, \cdots, y_k) \tag{20}$$

of $R\{x; y_1, \cdots, y_k\}$, writing its expansion in the form

$$a_1 x + \Sigma_{m,n_j} a_{m,n_1 \ldots n_k} x^m y_1^{n_1} \cdots y_k^{n_k} \tag{21}$$

In so writing it we have singled out the linear term in x, and the remaining sum extends over all those multi-indices for which

$$\{m; n_1, \cdots, n_k\} \neq \{1; 0, \cdots, 0\} \tag{22}$$

We next replace in (21) the quantity x by a series

$$\Sigma_p b_{p_1 \ldots p_k} y_1^{p_1} \cdots y_k^{p_k} \tag{23}$$

in $R\{y\}$, with undetermined coefficients $\{b_p\}$. This substitution replaces (21) by a series in $(y_1, \cdots, y_k)$ which we write in the form

$$F[x(y), y_j] = \Sigma_p c_{p_1 \ldots p_k} y_1^{p_1} \cdots y_k^{p_k} \tag{24}$$

We easily obtain an identity

$$c_{p_1 \cdots p_k} = a_1 \cdot b_{p_1 \cdots p_k} + \gamma_{p_1 \cdots p_k}(a;\, b_{q_1 \cdots q_k})$$

The first term is the contribution arising from the linear term a_1x whereas the remainder arises from the sum in (21). The remainder is a polynomial in some of the coefficients $\{a\}$ of the series (21) and some of the quantities $\{b_q\}$. However, and this is decisive, on the basis of (22) only those b_q can occur for which

$$(25) \qquad q_1 + \cdots + q_k < p_1 + \cdots + p_k$$

Furthermore, if we define the *weight* of the term $b_{q_1 \cdots q_k}$ to be $q_1 + \cdots + q_k$, then the weight of the polynomial $\gamma_{p_1 \cdots p_k}$ in the b's cannot exceed $p_1 + \cdots + p_k$. In particular, if $p_1 + \cdots + p_k = 1$, then by (25) the polynomial $\gamma_{p_1 \cdots p_k}$ is independent of the $\{b_q\}$. Hence if

$$(26) \qquad a_1 \neq 0$$

the series (24) will be 0, that is all its coefficients will vanish, if and only if

$$(27) \qquad b_{p_1 \cdots p_k} = -\frac{\gamma_{p_1 \cdots p_k}(a;\, b_q)}{a_1}$$

If the coefficients $\{a\}$ are given this is a system of equations for the computation of the coefficients b_p. We first compute directly those coefficients whose order is 1; this can be done since as we have seen $\gamma_{p_1 \cdots p_k}$ is independent of the $\{b_q\}$ for $p_1 + \cdots + p_k = 1$. Substituting these values into the right hand side of (27) we can next obtain those b_p whose order is 2, etc. Hence, if (26) holds, the equation

$$(28) \qquad F(x;\, y_1, \cdots, y_k) = 0$$

has one and only one solution in $R\{y_1, \cdots, y_k\}$, namely a series (23) whose coefficients are determined by the recurrence relations (27).

An analogous situation prevails for a *system* of equations

$$(29) \qquad F_\lambda(x_1, \cdots, x_l;\, y_1, \cdots, y_k) = 0, \qquad \lambda = 1, \cdots, l$$

$l \gtreqless k$. In each series F_λ we single out the linear terms in the variables x thus writing it in the form

$$(30) \qquad \Sigma_{\mu=1}^{l} a_\mu^\lambda x_\mu + \Sigma_{m;\,n} a^\lambda_{m_1 \cdots m_l;\, n_1 \cdots n_k} x_1^{m_1} \cdots x_l^{m_l} y_1^{n_1} \cdots y_k^{n_k}$$

and we replace each quantity x_μ, $\mu = 1, \cdots, l$, by a series

$$\Sigma_p b^\mu_{p_1 \cdots p_k} y_1^{p_1} \cdots y_k^{p_k}$$

with undetermined coefficients $\{b\}$ thus obtaining l series

$$\Sigma_p c^\lambda_{p_1 \cdots p_k} y_1^{p_1} \cdots y_k^{p_k} \tag{31}$$

We obtain a set of identities

$$c^\lambda_{p_1 \cdots p_k} = \Sigma^l_{\mu=1} a^\lambda_\mu \cdot b^\mu_{p_1 \cdots p_1} + \gamma^\lambda_{p_1 \cdots p_k}(a; b^\nu_{q_1 \cdots q_k}) \tag{32}$$

for which we again have the restriction (25), and in which again each $\gamma^\lambda_{p_1 \cdots p_k}$ is of weight at most $p_1 + \cdots + p_k$ in the b's. Putting (31) equal to zero is equivalent to putting

$$\Sigma^l_{\mu=1} a^\lambda_\mu b^\mu_{p_1 \cdots p_k} = - \gamma^\lambda_{p_1 \cdots p_k}(a; b^\nu_q)$$

and if we assume that the determinant

$$\Delta \equiv |a^\lambda_\mu|_{\lambda,\mu=1,\cdots,l} \tag{33}$$

is different from zero, this in turn is equivalent to a system of recurrence relations

$$b^\lambda_{p_1 \cdots p_k} = \frac{\delta^\lambda_{p_1 \cdots p_k}(a; b^\nu_q)}{\Delta} \tag{34}$$

where each δ^λ_p is again a polynomial in the a's and b's of weight at most $p_1 + \cdots + p_k$ in the b's and (25) is again valid. Thus on applying the same type of recursion argument applied previously for $l = 1$, we see that

$$\Delta \neq 0 \tag{35}$$

implies the existence of one and only one set of solutions of the system (29).

A case of special interest arises if, l being equal to k, each $F_j(x; y)$ is a difference $f_j(x) - g_j(y)$, $j = 1, \cdots, k$. In this case the coefficients a^j_μ as exhibited in (30) are simply the linear coefficients in the expansion

$$x'_j \equiv f_j(x_1, \cdots, x_k) = \Sigma a^j_\mu x_\mu + \text{(higher powers)} \tag{36}$$

In this case the quantity

$$\Delta \equiv |a^j_\mu|_{j,\mu=1,\cdots,k} \tag{37}$$

which depends only on the functions f_j, will be called the determinant of the system. Or rather, viewing (36) as an inner transformation, we will call Δ the *determinant of the transformation.* Whenever we denote the transformation by a single letter, such as T, we will also write specifically Δ_T. Returning now to our equations we obtain the following result. *If $f_j(x)$ and $g_j(y)$ are any k elements of $R\{x_1, \cdots, x_k\}$ and $R\{y_1, \cdots, y_k\}$ respectively, and if the determinant of the system f_j is different from zero, then the system of equations*

$$(38)\qquad f_j(x_1, \cdots, x_k) = g_j(y_1, \cdots, y_k), \qquad j = 1, \cdots, k$$

has one and only one solution

$$(39)\qquad x_j = \varphi_j(y_1, \cdots, y_k)$$

where the $\varphi_j(y)$ are elements of $R\{y_1, \cdots, y_k\}$. Denoting the transformation (36) by T, the transformation

$$(40)\qquad y'_j = g_j(y_1, \cdots, y_k)$$

by V, and the transformation (39) by U, we obtain the following result: *If T and V are arbitrary inner maps and $\Delta_T \neq 0$, then there exists a unique solution U of the system*

$$(41)\qquad TU = V$$

As a particular instance of this result choose for V the identity (19), denoting it by I. Then we see: if $\Delta_T \neq 0$ then there exists a right inverse, that is there exists an element U for which

$$(42)\qquad TU = I$$

If P, Q, R are three inner mappings of the form

$$(43)\qquad P: \quad x_j = a^j_1 y_1 + \cdots + a^j_k y_k + \text{(higher powers)}$$
$$(44)\qquad Q: \quad y_j = b^j_1 z_1 + \cdots + b^j_k z_k + \text{(higher powers)}$$
$$(45)\qquad R: \quad x_j = c^j_1 z_1 + \cdots + c^j_k z_k + \text{(higher powers)}$$

and if $PQ = R$, then

$$(46)\qquad c^j_\mu = \Sigma^k_{\nu=1} a^j_\nu b^\nu_\mu, \qquad j, \mu = 1, \cdots, k$$

In particular

$$(47)\qquad \Delta_{PQ} = \Delta_P \cdot \Delta_Q$$

Applying this to the case (42) we obtain

$$\Delta_T \cdot \Delta_U = 1$$

which implies that $\Delta_T \neq 0$ is not only a sufficient condition but also a necessary condition for the existence of a right inverse U. Now

multiply both sides of (42) by T on the right. Using (18) we obtain

$$T(UT) = (TU)T = I \cdot T = T = T \cdot I$$

However, if $TW = TW'$ and $\Delta_T \neq 0$ then by the uniqueness result proved for equation (41) we conclude that $W = W'$. Hence we conclude that

$$UT = I \tag{48}$$

Thus (42) implies (48). In other words: *an inner transformation T has a right inverse or left inverse if and only if its determinant is different from zero; the two inverses are always equal. This unique inverse will be denoted by T^{-1}.*

In keeping with customary usage we will call a transformation T *nonsingular* if it has an inverse, that is if $\Delta_T \neq 0$. It is easy to see that *all nonsingular transformations form a group.*

The simplest type of mappings are linear mappings. They have the form

$$x_j = a_1^j y_1 + \cdots + a_k^j y_k, \qquad j = 1, \cdots, k \tag{49}$$

They again form a semigroup, and the nonsingular ones a group. If (43) is any transformation P, the corresponding transformation (49), with all terms of higher order omitted, will be called its *linear part*, and it will be denoted by L_P or $L(P)$. By (46), if $PQ = R$, then $L_PL_Q = L_R$. Also $(L_P)^{-1} = L_{P^{-1}}$ for P nonsingular. Thus the transition from P to L_P is a homomorphism, i.e. a one-valued correspondence which preserves multiplicative relations. Under this homomorphism the image of a semigroup of (nonsingular) elements is again a (nonsingular) semigroup and the image of a group is a group. For an arbitrary family of transformations, several transformations may have the same linear part, that is the correspondence between transformation and linear part need not be one-to-one. If it happens to be one-to-one for a given family of transformations, that is if our homomorphism is a strict isomorphism, we will then say that the transformations involved are uniquely determined by their linear parts.

It is the purpose of the present chapter to discuss problems bearing on this unique determination.

§2. A Theorem Concerning the Circular Group

Given k coordinates $x_1, \cdots, x_k$, one of the simplest groups of transformations is the group

$$T(\theta): \qquad x_j' = e^{i\theta}x_j, \qquad j = 1, \cdots, k \tag{50}$$

It is defined for all real θ, and for each θ, it multiplies every coordinate x_j by the same factor $e^{i\theta}$ of absolute value unity. Obviously

$$T(\theta_1)T(\theta_2) = T(\theta_1 + \theta_2), \qquad T(\theta)^{-1} = T(-\theta)$$

and

$$T(\theta + 2\pi) = T(\theta)$$

We will call this group the *circular group*.

Theorem 1. *If a semigroup S of inner transformations is uniquely determined by its linear parts, and if it includes the circular group* (50), *then each element of S is linear.*

Proof. Let

$$A: \qquad x_j = \Sigma_n a^j_{n_1\cdots n_k} y_1^{n_1} \cdots y_k^{n_k}, \qquad j = 1, \cdots, k$$

be any element of S. The transformation $T(\theta)A$ has the form

$$x_j = \Sigma_n e^{i\theta} a^j_{n_1\cdots n_k} y_1^{n_1} \cdots y_k^{n_k}$$

and $AT(\theta)$ has the form

$$x_j = \Sigma_n a^j_{n_1\cdots n_k} (e^{i\theta}y_1)^{n_1} \cdots (e^{i\theta}y_k)^{n_k}$$

that is,

$$x_j = \Sigma_n e^{i(n_1+\cdots+n_k)\theta} a^j_{n_1\cdots n_k} y_1^{n_1} \cdots y_k^{n_k}$$

Since S is a semigroup, both transformations $T(\theta)A$ and $AT(\theta)$ belong to S. Their coefficients are respectively

$$e^{i\theta} a^j_{n_1\cdots n_k}, \qquad e^{i(n_1+\cdots+n_k)\theta} a^j_{n_1\cdots n_k}$$

If $n_1 + \cdots + n_k = 1$, the exponential factors coincide and thus the mappings have the same linear parts. By hypothesis, being members of S, they must be equal in all other coefficients. However, if $n_1 + \cdots + n_k > 1$, the exponential factors cannot coincide for all values of θ, and hence all coefficients of order greater than unity must be zero. Q.E.D.

§3. Topology of Formal Power-series

We consider for fixed k and l all transformations (5). The general coefficient will be denoted by

$$(51) \qquad a^\lambda_{n_1\cdots n_k}$$

If $\{T(\alpha)\}$ is any set T of such transformations, where α is any index which tells the elements of T apart, then the corresponding coefficients will be denoted by $a^\lambda_{n_1\cdots n_k}(\alpha)$. We now introduce a limit topology

in the space of all transformations (5). Since, on the one hand, each transformation is a multi-sequence of numbers (51) for $\lambda = 1, \cdots, l$ and $n_1, \cdots, n_k = 0, 1, 2, \cdots$, and since, on the other hand, we are not hampered by the requirement that the series (5) have a domain of convergence, it is very natural (and, as we shall see, very appropriate) to introduce the ordinary weak topology based upon convergence for each coefficient separately. Thus we call a sequence $\{T(s)\}$, $s = 1, 2, \cdots$ (*weakly*) *convergent* if

$$\lim_{s \to \infty} a^{\lambda}_{n_1 \cdots n_k}(s) = a^{\lambda}_{n_1 \cdots n_k}$$

exists for every $(\lambda, n_1, \cdots, n_k)$. The transformation T whose coefficients are the latter limits is the (*weak*) *limit* of the sequence $\{T(s)\}$. The set T is (weakly) *closed* if it contains the limit of all of its convergent sequences. The set T is (weakly) *bounded* if each coefficient is bounded, that is if there exist positive numbers $A^{\lambda}_{n_1 \cdots n_k}$ such that

$$|a^{\lambda}_{n_1 \cdots n_k}(\alpha)| \leq A^{\lambda}_{n_1 \cdots n_k} \tag{52}$$

The set T is called (weakly) *compact* if it is bounded and closed. By a known argument, the set T is bounded if and only if every sequence $\{T(s)\}$ in T contains a subsequence $\{T(s_\nu)\}$ which is convergent, and T is compact if and only if the limits of these subsequences also belong to T.

Now let $\{T(\alpha)\}$ be a set of transformations from x-coordinates into y-coordinates and U a fixed transformation from y-coordinates to z-coordinates. In the transformation UT each coefficient is a *polynomial* of the coefficients of T and U. This shows readily that if $\{T(\alpha)\}$ is bounded, closed, convergent, etc. then the same is true for $\{UT(\alpha)\}$. Similarly if we multiply $T(\alpha)$ on both sides by fixed transformations U and V, then the resulting family $\{UT(\alpha)V\}$ has the same convergence properties as $\{T(\alpha)\}$. Now let $\{T(\alpha)\}$ be a set of *inner* transformations in k $(= l)$ coordinates, and let U be any other fixed inner transformation (also in k coordinates) which has an inverse. Then we can form the new family

$$T'(\alpha) = U^{-1}T(\alpha)U \tag{53}$$

The transition from $\{T(\alpha)\}$ to $\{T'(\alpha)\}$ is called a *similarity*. Similarity preserves group operations, since

$$U^{-1}T(\alpha)U \cdot U^{-1}T(\beta)U = U^{-1}T(\alpha)T(\beta)U$$

and

$$U^{-1}IU = I$$

and as we have just seen, it also preserves topological properties. Furthermore, similarity is an invertible process, since, putting $V = U^{-1}$, we have $T(\alpha) = V^{-1}T'(\alpha)V$. If we write the transformations $T(\alpha)$, U, and U^{-1} in the forms

$$T(\alpha): \quad y_j = f_j(x); \qquad U: \quad x_j = \varphi_j(x'); \qquad U^{-1}: \quad y'_j = \psi_j(y); \qquad j = 1, \cdots, k$$

then the new transformation (53) has the form

$$T'(\alpha): \qquad y'_j = \psi_j(f(\varphi(x')))$$

Thus it is appropriate to interpret relation (53) in the following way. $T(\alpha)$ is a transformation carrying the point with coordinates (x) into the point with coordinates (y); U is a universal change of the coordinate system, affecting (x) and (y) simultaneously; and $T'(\alpha)$ expresses the original transformation $T(\alpha)$ in terms of the changed coordinates.

Finally, assume that $T(\alpha)$ is nonsingular and consider its inverse $T(\alpha)^{-1}$. If $a_n^\lambda(\alpha)$ is the general coefficient of $T(\alpha)$, and $b_n^\lambda(\alpha)$ is the general coefficient of $T(\alpha)^{-1}$ and $\Delta(\alpha)$ is the determinant of $T(\alpha)$, then on the basis of formula (34) each $b_n^\lambda(\alpha)$ can be expressed as a fraction whose numerator is a polynomial in the coefficients $a_\nu^\lambda(\alpha)$ and whose denominator is $[\Delta(\alpha)]^{n_1+\cdots+n_k}$. Hence we obtain the following result.

Theorem 2. *If $\{T(\alpha)\}$ is a (weakly) bounded set (i.e. if (52) holds), and if there exists a positive constant c such that $|\Delta(\alpha)| \geq c$, then the set $\{T(\alpha)^{-1}\}$ is likewise (weakly) bounded.*

This theorem will be applied in due course.

§4. A Uniqueness Theorem of Henri Cartan

Theorem 3. *If T is an inner transformation whose linear part is the identity, that is if T has the form*

$$x'_j = x_j + \text{(higher powers)} \tag{54}$$

and if the set of iterated transformations $\{T, T^2, T^3, \cdots\}$ is (weakly) bounded, then T actually is the identity.

Proof. We write (54) explicitly in the form

$$T: \qquad x'_j = x_j + A_r^j(x) + A_{r+1}^j(x) + \cdots \tag{55}$$

According to our previous convention (cf. section 1), $A_p^j(x)$ is a homogeneous polynomial of order p; the index r in (55) is some integer

≥ 2; and writing T in the form (55) implies, if $r > 2$, that all the coefficients of order > 1 and $< r$ are known to vanish for $j = 1, \cdots, k$. Iterating (55) we obtain for T^2 the expansion

$$T^2: \qquad x_j'' = x_j' + A_r^j(x') + \cdots = x_j + 2A_r^j(x) + \cdots$$

and in general

$$(56) \quad T^s: \qquad x_j^{(s)} = x_j + sA_r^j(x) + \cdots$$

If $a^j_{r_1 \cdots r_k}$, $r_1 + \cdots + r_k = r$, is any coefficient occurring in (55), then by our hypothesis on the set $\{T, T^2, T^3, \cdots\}$ there exists a number M which is independent of s such that

$$(57) \qquad s|a^\lambda_{r_1 \cdots r_k}| \leq M$$

Since (57) holds for each $s = 1, 2, 3, \cdots$, we conclude that $a_r^j = 0$, and thus that $A_r^j(x) \equiv 0$; and so we find that (54) may not contain any nonvanishing coefficients of order greater than 1.

This is Cartan's uniqueness theorem. It leads to the following conclusion the first half of which was pointed out by Carathéodory.

Theorem 4. *If S is a bounded group of inner transformations then the elements of S are uniquely determined by their linear parts.*

More precisely, if $T(\alpha)$, $T(\beta)$, $\cdots$ are elements of S, if we write the expansion of $T(\alpha)$ in the form

$$(58) \quad y_j = a_1^j(\alpha)x_1 + \cdots + a_k^j(\alpha)x_k + \Sigma_{n_1 + \cdots + n_k \geq 2} a^j_{n_1 \cdots n_k}(\alpha) x_1^{n_1} \cdots x_k^{n_k}$$

and if $(j, n_1, \cdots, n_k)$ is any fixed multi-index, then corresponding to any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$(59) \qquad \Sigma^k_{\rho,\sigma=1} |a^\rho_\sigma(\alpha) - a^\rho_\sigma(\beta)| \leq \delta$$

implies

$$(60) \qquad |a^j_{n_1 \cdots n_k}(\alpha) - a^j_{n_1 \cdots n_k}(\beta)| \leq \epsilon$$

Proof. If U and V belong to S, then so does $T = UV^{-1}$, hence $\{T, T^2, \cdots\}$ is a bounded set. If $L(U)$ and $L(V)$ are the linear parts of U and V then, as we saw at the end of section 1,

$$L(T) = L(U)L(V)^{-1}$$

Hence if $L(U) = L(V)$, then $L(T)$ is the identity and by Theorem 3 T itself is the identity. Thus $UV^{-1} = I$, or $U = V$. Thus the elements of S are uniquely determined by their linear parts.

For the proof of the second half of the theorem, we first assume that the group S is not only bounded but also closed, that is, compact. Under this assumption if our assertion were false there would exist a multi-index $(j, n_1, \cdots, n_k)$, a $\delta_0 > 0$, and two sequences $\{T(\alpha_s)\}$, $\{T(\beta_s)\}$, $s = 1, 2, \cdots$, of group elements such that simultaneously

$$(61) \qquad \left|a^j_{n_1 \cdots n_k}(\alpha_s) - a^j_{n_1 \cdots n_k}(\beta_s)\right| \geq \delta_0$$

and

$$(62) \qquad \Sigma^k_{\rho,\sigma=1}\left|a^\rho_\sigma(\alpha_s) - a^\rho_\sigma(\beta_s)\right| \leq \frac{1}{s}$$

Since S is bounded, there exists a subsequence $\{s_p\}$ of the integers $\{s\}$ such that the sequences $\{T(\alpha_{s_p})\}$ and $\{T(\beta_{s_p})\}$ are both (weakly) convergent. Denoting their limits by $T(\alpha)$ and $T(\beta)$, we also know that $T(\alpha)$ and $T(\beta)$ belong to S, since S temporarily is assumed to be closed. From (61) and (62) we now conclude that

$$(63) \qquad \left|a^j_{n_1 \cdots n_k}(\alpha) - a^j_{n_1 \cdots n_k}(\beta)\right| \geq \delta_0$$

and

$$(64) \qquad \Sigma^k_{\rho,\sigma=1}\left|a^\rho_\sigma(\alpha) - a^\rho_\sigma(\beta)\right| = 0$$

Relation (64) states that $T(\alpha)$ and $T(\beta)$ have the same linear parts, whereas (63) implies that $T(\alpha)$ and $T(\beta)$ are not equal. This contradicts the first half of the theorem, and this proves the second half of our theorem under the additional assumption that S is not only bounded but also closed.

Now if S is not closed, we take its set theoretical closure $\bar{S}$. It consists of all transformations $T(\bar{\alpha})$, $T(\bar{\beta})$, etc. which are limits of convergent sequences $\{T(\alpha_s)\}$, $\{T(\beta_s)\}$, etc. where $T(\alpha_s)$, $T(\beta_s)$ are elements of S, and repetitions in the sequences are allowed. Expanding slightly the argument which was used in section 3 we see that the sequence of the products $T(\alpha_s)T(\beta_s)$ converges towards the product of the limits, that is, towards $T(\bar{\alpha})T(\bar{\beta})$. Furthermore, we see that this product is independent of the special approximating sequences used, and that for three convergent sequences the associative relation

$$(T(\alpha_s)T(\beta_s))T(\gamma_s) = T(\alpha_s)(T(\beta_s)T(\gamma_s))$$

in S yields the associative law in $\bar{S}$. Finally let $T(\bar{\alpha})$ be any element in $\bar{S}$, let $\{T(\alpha_s)\}$ be a convergent sequence in S with limit $T(\bar{\alpha})$, and

for each s denote by $T(\beta_s)$ the inverse of $T(\alpha_s)$, so that

$$T(\alpha_s)T(\beta_s) = I \tag{65}$$

Since S is bounded there exists a subsequence $\{\beta_{s_\nu}\}$ of $\{\beta_s\}$ such that $T(\beta_{s_\nu}) \to T(\bar{\beta})$. Thus from (65) we derive the limit relation

$$T(\bar{\alpha})T(\bar{\beta}) = I$$

and thus every element $T(\bar{\alpha})$ of $\bar{S}$ possesses an inverse $T(\bar{\beta})$ which likewise belongs to $\bar{S}$. Thus we see that the closure $\bar{S}$ of any bounded group S is again a group. Now $\bar{S}$ is compact, and the second half of Theorem 4 has been proved to apply to compact groups. Hence Theorem 4 applies to $\bar{S}$ and so a fortiori it applies to S itself.

For later use we include the following theorem.

Theorem 5. *If T is an inner transformation with $|\Delta_T| = 1$, and if the semigroup $\{T, T^2, T^3, \cdots\}$ is bounded, then there exists a sequence $\{T^{p_1}, T^{p_2}, T^{p_3}, \cdots\}$, $1 \leq p_1 \leq p_2 \leq \cdots$, which converges to the identity.*

In fact by Carathéodory's theorem (Theorem 4), it suffices to find exponents $\{p_s\}$ for which the linear parts converge to the identity. Also it is easily seen that $L(T^p) = [L(T)]^p$ (compare equation (46)). In view of these two facts we need only prove the theorem for a linear transformation $T = L$. Due to the boundedness of L^n, there exist exponents $n_1 < n_2 < \cdots$, such that if we put

$$L^{n_s}: \qquad x'_j = \Sigma_{\nu=1}^k a_{j\nu}(s)x_\nu$$

the limits

$$\lim_{s\to\infty} a_{j\nu}(s)$$

exist for $j, \nu = 1, \cdots, k$. Since $|\Delta_{L^n}| = 1$, the inverse transformations

$$L^{-n_s}: \qquad x'_j = \Sigma_{\nu=1}^k b_{j\nu}(s)x_\nu$$

are also convergent. Therefore the sequence

$$L^{n_{s+1}} \cdot L^{-n_s} = L^{n_{s+1}-n_s}$$

is convergent, and indeed with limit the product of the limit of $L^{n_{s+1}}$ by the limit of L^{-n_s}, that is with limit the identity. Thus $p_s = n_{s+1} - n_s$ will satisfy the theorem.

§5. Bounded Groups of Transformations

In the present section we will deal in substance with linear (inner) transformations

$$T: \qquad y_j = \Sigma_{\mu=1}^k a_\mu^j x_\mu, \qquad j = 1, \cdots, k$$

The determinant $\Delta = \Delta_T$ is a polynomial in the a^j_μ. Hence, if a set $\{T(\alpha)\}$ is (weakly) bounded according to our definition in section 3, the numbers $\Delta(\alpha)$ are bounded. Suppose in particular that T belongs to a bounded semigroup. Then the set of iterates $\{T, T^2, T^3, \cdots\}$ is bounded. However,

$$\Delta_{T^s} = (\Delta_T)^s = \Delta^s$$

and the set of complex numbers Δ^s, $s = 1, 2, \cdots$ can be bounded only if $|\Delta| \leq 1$. If T belongs to a bounded group then the set $\{T^{-s}\}$ must also be bounded. In this case we must therefore also have $|\Delta^{-1}| \leq 1$, and since both inequalities hold simultaneously we have $|\Delta| = 1$ for T a member of a bounded group. Actually, more precise statements are available. It is known from linear algebra that given T there exists a nonsingular linear transformation S such that the similar transformation

$$U = S^{-1}TS \tag{66}$$

has special features. In fact, writing U in the form

$$\eta_j = \Sigma^k_{\mu=1} b^j_\mu \xi_\mu \tag{67}$$

it is possible to choose S in such a way that in the matrix $||b^j_\mu||$ all terms under the main diagonal shall have the value zero, that is

$$b^j_\mu = 0 \qquad \text{for} \qquad \mu < j \tag{68}$$

If the matrix b is thus specialized the diagonal terms

$$\lambda_1 = b^1_1, \qquad \lambda_2 = b^2_2, \qquad \cdots, \qquad \lambda_k = b^k_k$$

are uniquely determined but for their order. They are the characteristic roots of the original matrix and are invariant under similarities. The matrix of each iterate U^s has again all zeros under the main diagonal, and its characteristic roots are the s-th powers of those of U, namely $\lambda^s_1, \cdots, \lambda^s_k$. Also, since the terms under the main diagonal vanish, the determinant $|b^j_\mu|$ for U has the value $\lambda_1 \cdots \lambda_k$. But by (66) the determinant Δ of T is the same as the determinant $|b^j_\mu|$ of U. Hence the determinant Δ has the value $\lambda_1 \cdots \lambda_k$. Now in section 3 we have seen that if $\{T^s\}$ is bounded then so is $\{U^s\}$. Hence each term in the matrix of U^s is bounded in s, in particular the powers λ^s_μ are bounded for each $\mu = 1, \cdots, k$. This implies

$$|\lambda_1| \leq 1, \qquad \cdots, \qquad |\lambda_k| \leq 1 \tag{69}$$

and this set of inequalities is much more explicit than $|\Delta| \leq 1$ which on

the face of it, implies only that $|\lambda_1 \cdots \lambda_k| \leq 1$. Once (69) is established, the equality $|\Delta| = 1$ can hold only if

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_k| = 1 \tag{70}$$

All of this can be applied to the linear part of any inner transformation rather than merely to a linear transformation.

Theorem 6. *If T is any element of a bounded semigroup of inner transformations then the determinant Δ_T satisfies the inequality $|\Delta_T| \leq 1$ and the characteristic roots of its linear part satisfy relation* (69).

If T belongs to a bounded group (of inner transformations) then $|\Delta_T| = 1$ and the characteristic roots satisfy the sharper relation (70).

If the set $\{T, T^2, \cdots\}$ is bounded and if (70) *holds, (or even if only $|\Delta| = 1$ holds), then $\{T^{-1}, T^{-2}, \cdots\}$ is also bounded and thus T is an element of the bounded group $\{T^s\}$, $s = 0, \pm 1, \pm 2, \cdots$.*

Proof. The semigroup $\{T, T^2, T^3, \cdots\}$ is contained in any semigroup including T, and thus the first half of the theorem only summarizes the preceding remarks. The second half of the theorem is a consequence of Theorem 2, since (70) implies $|\Delta| = 1$.

If T is linear, then by choosing S more judiciously in (66) we can obtain not only the relations (68) but also the additional relations

$$b^j_\mu = 0 \qquad \text{for} \qquad j + 1 < \mu$$

Thus the only coefficients in (67) which may be different from zero are

$$\lambda_j = b^j_j, \qquad \gamma_j = b^j_{j+1}, \qquad j = 1, \cdots, k$$

However, as is well known from linear algebra, each γ_j must be either 1 or 0, and whenever $\gamma_j = 1$ for some index j, then for the same index j, $\lambda_j = \lambda_{j+1}$. Having brought about this canonical form of the matrix $\|b^j_\nu\|$, we now form the iterated transformation U^{s+1} for any fixed positive integer s, and we denote its matrix by $\|c^j_\nu\|$. The latter is the familiar $(s + 1)$-th power of the matrix $\|b^j_\nu\|$, and hence by an elementary formula of matrix theory

$$c^p_q = \Sigma^k_{\nu_1=1} \cdots \Sigma^k_{\nu_s=1} b^p_{\nu_1} b^{\nu_1}_{\nu_2} \cdots b^{\nu_{s-1}}_{\nu_s} b^{\nu_s}_q$$

In our special case, owing to (68), we may restrict ourselves for $q > p$, to such combinations $(\nu_1, \cdots, \nu_s)$ for which

$$p \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_s \leq q \tag{71}$$

Now choose a fixed index j, and put $p = j$, $q = j + 1$. This will allow in (71) only one actual inequality at a time. Thus the coefficient

c^i_{j+1} has the value

$$\gamma_j\lambda^s_{j+1} + \lambda^1_j\gamma_j\lambda^{s-1}_{j+1} + \lambda^2_j\gamma_j\lambda^{s-2}_{j+1} + \cdots + \lambda^{s-1}_j\gamma_j\lambda_{j+1} + \lambda^s_j\gamma_j$$

Now if $\gamma_j = 1$, then as we remarked earlier in this paragraph, $\lambda_{j+1} = \lambda_j$ and so we have $c^i_{j+1} = (s+1)\lambda^s_j$. If now $|\lambda_j| = 1$, then $|c^i_{j+1}| = s + 1$, and this is not bounded as $s \to \infty$. Hence if all positive and negative powers of U are bounded then no coefficient $\gamma_j = b^i_{j+1}$ can be different from zero. Hence we have the following theorem.

Theorem 7. *If T is an element of a bounded group of inner transformations then there exists a linear change of variables S, which may depend on T, such that similar transformation $U = S^{-1}TS$ has the canonical form*

$$U\colon\quad y_j = \lambda_j x_j + (\text{higher powers}), \qquad j = 1, \cdots, k \tag{72}$$

with $|\lambda_1| = \cdots = |\lambda_k| = 1.$

§6. Bounded Groups of Transformations. A Generalization of Cartan's Uniqueness Theorem

Theorem 8. *Any bounded group of inner transformations $\{T(\alpha)\}$ is similar to the group of its linear parts $\{L(T(\alpha))\}$: there exists a transformation S whose linear part is the identity*

$$S\colon\quad x'_j = x_j + (\text{higher powers}), \qquad j = 1, \cdots, k \tag{73}$$

such that for all α

$$T(\alpha) = S^{-1}L(T(\alpha))S$$

The proof will be based upon Theorem 3 by way of Theorem 4. However, once established, Theorem 8 includes Theorem 3 in the following peculiar manner. If T is any transformation whose linear part is the identity and for which the set $\{T, T^2, \cdots\}$ is bounded then (see Theorem 6) the cyclic group $\{T^s\}$, $s = 0, \pm 1, \pm 2, \cdots$, is bounded. Hence by Theorem 8, there exists a nonsingular S such that $T = S^{-1}L(T)S$. But $L(T) = I$, hence $T = I$, as asserted in Theorem 3.

In the course of the proof of Theorem 8 it will be necessary to form additive combinations of transformations. If $y_j = f_j(x)$ and $y_j = g_j(x)$ are two transformations T and U and a, b are complex numbers, then $aT + bU$ is to be the transformation $y_j = af_j(x) + bg_j(x)$. If V is a third transformation $z_j = h_j(y)$, then the distributive law in the form

$$(aT + bU)V = aTV + bUV$$

always holds while in general it does *not* hold in the form $V(aT + bU) = aVT + bVU$. However it *does* hold even in the second form if V is linear. Thus for any linear transformation L, and arbitrary transformations V, $T_1, \cdots, T_p$, and arbitrary complex numbers $c_1, \cdots, c_p$, we have

$$(74)\quad \begin{aligned}(c_1T_1 + \cdots + c_pT_p) \cdot V &= c_1T_1 \cdot V + \cdots + c_pT_p \cdot V,\\ L \cdot (c_1T_1 + \cdots + c_pT_p) &= c_1L \cdot T_1 + \cdots + c_pL \cdot T_p\end{aligned}$$

On the other hand, as we have seen in section 3, the product UW is continuous in each factor, as a matter of fact even in both factors. Thus

$$(75)\quad \begin{aligned}\lim_{p\to\infty} (U_p \cdot V) &= (\lim_{p\to\infty} U_p) \cdot V\\ \lim_{p\to\infty} (L \cdot U_p) &= L \cdot (\lim_{p\to\infty} U_p)\end{aligned}$$

provided the limit on the right hand side exists. Relation (75) suggests the possibility of extending (74) from sums to integrals.

Assume that a point set G is given, say the interval $0 \leq \alpha \leq 1$, and assume that an inner transformation $T(\alpha)$ is defined for each α in G,

$$(76)\quad T(\alpha):\quad y_j = \Sigma^{\infty}_{n_1,\cdots,n_k=0} a^j_{n_1\cdots n_k}(\alpha)x_1^{n_1} \cdots x_k^{n_k},\qquad j = 1, \cdots, k, \qquad n_1 + \cdots + n_k > 0$$

We further assume that for certain sets A of G a finitely additive measure $\mu(A)$ is defined for which the total set G has measure 1; for instance, if G is the interval (0, 1) the ordinary Euclidean length of subintervals and finite combinations of intervals is such a measure. Finally we assume that each coefficient

$$a(\alpha) = a^j_{n_1\cdots n_k}(\alpha)$$

is (bounded and) Riemann integrable in relation to the measure just assumed. Denoting the value of the integral by

$$\int_G a(\alpha)d\mu(\alpha)$$

integrability of $a(\alpha)$ means that corresponding to each $\epsilon > 0$ there exists a partitioning of G into sets $A_1, \cdots, A_s$ such that for any points α_r of A_r, $r = 1, \cdots, s$,

$$\left|\int_G a(\alpha)d\mu(\alpha) - \Sigma^s_{r=1}a(\alpha_r)\mu(A_r)\right| \leq \epsilon$$

For the ordinary Euclidean length on the interval (0, 1) this is the ordinary Riemann integral, and more generally, for any monotonely increasing (nondecreasing) function $\mu(\alpha)$ with $\mu(0) = 0$, $\mu(1) = 1$,

this is the familiar Stieltjes integral $\int_0^1 a(\alpha)\, d\mu(\alpha)$. In accordance with our concept of weak convergence (see section 3) we now define the integral

$$\int_G T(\alpha) d\mu(\alpha) \tag{77}$$

as the transformation whose coefficients are

$$\int_G a^j_{n_1 \cdots n_k}(\alpha) d\mu(\alpha) \tag{78}$$

Since the total measure of G is 1, it follows that if the linear part of each $T(\alpha)$ is the identity then the linear part of (77) is the identity.

If L is any fixed linear transformation

$$L: \qquad z_j = \Sigma_{\mu=1}^k b^j_\mu y_\mu, \qquad j = 1, \cdots, k$$

then the coefficients of $L \cdot T(\alpha)$ are

$$c^j_{n_1 \cdots n_k} = \Sigma_{\mu=1}^k b^j_\mu a^\mu_{n_1 \cdots n_k}(\alpha)$$

and clearly these are again integrable. Hence we can form

$$\int_G L \cdot T(\alpha) d\mu(\alpha) \tag{79}$$

We assert that

$$\int_G L \cdot T(\alpha) d\mu(\alpha) = L \cdot \int_G T(\alpha) d\mu(\alpha) \tag{80}$$

In fact, in the case of ordinary measure in the interval (0, 1) we approximate to (77) by values

$$T_s = \frac{1}{s}\left[T(0) + T\left(\frac{1}{s}\right) + T\left(\frac{2}{s}\right) + \cdots + T\left(\frac{s-1}{s}\right)\right]$$

On the one hand, by (74)

$$\frac{1}{s}\,\Sigma_{r=0}^{s-1} L \cdot T\left(\frac{r}{s}\right) = L \cdot T_s \tag{81}$$

On the other hand, by the definition of the Riemann integral each coefficient of T_s converges towards (78) and hence T_s converges towards (77), and thus by (75) the right member of (81) converges towards the right member of (80). Similarily the left member of (81) converges towards the left member of (80), and thus (80) follows from (81). This reasoning also applies to the most general measure $\mu(A)$ and thus (80) always holds.

As a companion to (80) we will also need the relation

$$\int_G T(\alpha) \cdot V d\mu(\alpha) = \left(\int_G T(\alpha) d\mu(\alpha)\right) \cdot V \tag{82}$$

where V is any fixed transformation. The integrability of $T(\alpha) \cdot V$ follows immediately from the integrability of $T(\alpha)$. The remainder of the proof is essentially the same as for (80), being based again upon (75), this time the first part. We omit the details.

We now turn to the proof of Theorem 8. Writing the linear part $L(T(\alpha))$ of $T(\alpha)$ in the form

$$y_j = \Sigma_{\mu=1}^{k} a_\mu^j(\alpha) x_\mu$$

and remembering that by Theorem 4, $T(\alpha)$ is uniquely determined by $L(T(\alpha))$, we see that $T(\alpha)$ is uniquely determined by the k^2 complex numbers

$$a_\mu^j \equiv a_\mu^j(\alpha), \qquad j,\ \mu = 1, \cdots, k$$

Taking the real and imaginary part of each a_μ^j, we obtain $2k^2$ numbers which we interpret as the coordinates of a point in the Euclidean space E of $2k^2$ dimensions. Every element α of G can therefore (be represented by, and hence for our purposes) be identified with a point in E, and G itself with a bounded point-set in E. Also, since

$$L(T(\alpha) \cdot T(\beta)) = L(T(\alpha)) \cdot L(T(\beta)) \tag{83}$$

the group product $\beta\alpha$ of any two elements α, β of G is given by

$$a_\mu^j(\beta\alpha) = \Sigma_{\nu=1}^{k} a_\nu^j(\beta) a_\mu^\nu(\alpha)$$

Thus in the ordinary Euclidean topology of E, the group product $\beta\alpha$ is a continuous function of β and α.

Now since G is bounded, there exists an additive measure $\mu(A)$ on G with the following properties, among others:

(i) the measure of the total set G is 1,

(ii) the measure is group invariant, and

(iii) every function $f(\alpha)$ on G which is uniformly continuous on G (in the topology of E) is integrable.

For our purpose the best way of expressing property (ii) is as follows: If $f(\alpha)$ is any integrable function and β is any fixed element, then

$$\int_G f(\beta\alpha) d\mu(\alpha) = \int_G f(\alpha\beta) d\mu(\alpha) = \int_G f(\alpha) d\mu(\alpha) = \int_G f(\alpha^{-1}) d\mu(\alpha) \tag{84}$$

Now we see the significance of the assertion of the more precise part of Theorem 4. It states that each coefficient of $T(\alpha)$ is uniformly continuous on G. Hence (by property (iii)) each coefficient of $T(\alpha)$ is integrable relative to our measure. Furthermore since a polynomial of integrable functions is integrable, and since $a(\alpha^{-1})$ is integrable if $a(\alpha)$ is (cf. (84)), it follows that the transformation $L(T(\alpha^{-1})) \cdot T(\alpha)$

is integrable. The linear part of this transformation is the identity since

$$L(L(T(\alpha^{-1})) \cdot T(\alpha)) = L(T(\alpha^{-1})) \cdot L(T(\alpha)) = L(T(\alpha^{-1}) \cdot T(\alpha)) = L(I) = I$$

We now introduce the transformation

$$S = \int_G L(T(\alpha^{-1})) \cdot T(\alpha) d\mu(\alpha) \tag{85}$$

We first note that the linear part of S is the identity, this follows at once from the preceding relation and property (i) of our measure. Next let β be any fixed element of G. By (80),

$$L(T(\beta)) \cdot S = \int_G L(T(\beta)) \cdot (L(T(\alpha^{-1})) \cdot T(\alpha)) d\mu(\alpha)$$

and by the associative law (18) and the relation (83) this is equal to

$$\begin{aligned}\int_G L(T(\beta) \cdot T(\alpha^{-1})) \cdot T(\alpha) d\mu(\alpha) &= \int_G L(T(\beta\alpha^{-1})) \cdot T(\alpha) d\mu(\alpha) \\ &= \int_G L(T(\beta\alpha^{-1})) \cdot T(\alpha\beta^{-1} \cdot \beta) d\mu(\alpha) \\ &= \int_G L(T(\beta\alpha^{-1})) T(\alpha\beta^{-1}) \cdot T(\beta) d\mu(\alpha)\end{aligned}$$

Finally, on account of (82), this is equal to

$$[\int_G L(T(\beta\alpha^{-1})) \cdot T(\alpha\beta^{-1}) d\mu(\alpha)] \cdot T(\beta)$$

Now the integrand $L(T(\beta\alpha^{-1}) \cdot T(\alpha\beta^{-1})$ arises from the integrand $L(T(\alpha^{-1})) \cdot T(\alpha)$ on replacing α by $\alpha\beta^{-1}$. By (84), this does not alter the value of the integral, and hence,

$$L(T(\beta)) \cdot S = S \cdot T(\beta)$$

which is the assertion of Theorem 8.

§7. A Theorem of Behnke and Peschl

We consider any inner transformation

$$U\colon\quad x'_j = g_j(x_1, \cdots, x_k) \equiv \Sigma_{n=n_j}^{\infty} A_n^j(x), \qquad j = 1, \cdots, k \tag{86}$$

with no restriction on the values of the smallest degrees n_j occurring in (86). We only assume that the determinant

$$J(x_1, \cdots, x_k) = \left|\frac{\partial A_{n_j}^j}{\partial x_\mu}\right|_{j,\,\mu=1,\ldots,k}$$

is not identically zero,

$$J(x_1, \cdots, x_k) \not\equiv 0 \tag{87}$$

Such a transformation will be called *nondegenerate*.

Behnke and Peschl have proved the following beautiful generalization of Theorem 3 (Cartan).

Theorem 9. *If T is an inner transformation of the form*

$$T: \qquad x'_j = x_j + \text{(higher powers)}$$

and if there exists a nondegenerate transformation U such that the set of transformations $\{UT, UT^2, UT^3, \cdots\}$ is (weakly) bounded, then the transformation T is the identity.

We will handle this theorem by showing that if the set $\{UT, UT^2, \cdots\}$ is bounded then so is the set $\{T, T^2, T^3, \cdots\}$ itself, and thus we will ultimately reduce Theorem 9 to Theorem 3. As a matter of fact, we will prove the following more general theorem.

Theorem 10. *If a set of transformations $\{T(\alpha)\}$ is such that their linear parts $L(T(\alpha))$ are bounded and their determinants are bounded away from zero,*

$$|\Delta_{T(\alpha)}| \geq \delta > 0$$

and if there exists a nondegenerate transformation U for which the set of transformations $\{UT(\alpha)\}$ is bounded, then the set $\{T(\alpha)\}$ itself is bounded.

In particular, if the linear parts of a group $\{T(\alpha)\}$ are bounded, and if $\{UT(\alpha)\}$ is bounded, then $\{T(\alpha)\}$ itself is bounded.

We will first state some simple properties of polynomials,

$$P(x_1, \cdots, x_k) = \Sigma_n a_{n_1 \ldots n_k} x_1^{n_1} \cdots x_k^{n_k}$$

whose degrees do not exceed a fixed integer m.

We will view these polynomials both as formal series and as functions of their complex variables. Since their degrees are bounded, a set of such polynomials is bounded if and only if their "norms"

$$(88) \qquad ||P|| = \Sigma_n |a_{n_1 \ldots n_k}|$$

are bounded.

Polynomials in one complex variable,

$$P(x) = a_0 + a_1 x + \cdots + a_m x^m, \qquad (m \text{ fixed})$$

are a compact family in a strong sense. If a neighborhood $|x - x^0| < r$ and a constant M are given, then the set of polynomials for which $|P(x)| \leq M$ in the neighborhood are bounded in norm. By induction on the dimension k this can be extended to several variables. If a neighborhood

$$(89) \qquad |x_\nu - x_\nu^0| < r, \qquad \nu = 1, \cdots, k$$

and a constant M are given, then the set of polynomials for which $|P(x_1, \cdots, x_k)| \leq M$ is a bounded set in norm. The converse of this being true, and boundedness in norm and weak boundedness being obviously equivalent for polynomials, it follows that the following three concepts are equivalent for polynomials, boundedness in norm (in the sense of (88)), weak boundedness (in the sense of section 3), and boundedness in the sense that for every neighborhood (89), there is a constant $M(x^0, r)$ such that $|P(x_1, \cdots, x_k)| \leq M$ in that neighborhood.

Now, consider a system of relations

$$(90) \qquad \Sigma_{\nu=1}^k \Delta_{j\nu}(x) P_\nu(x) = B_j(x), \qquad j = 1, \cdots, k$$

among polynomials $\Delta_{j\nu}$, P_ν, B_j, each of degree $\leq m$ in $x_1, \cdots, x_k$. Suppose that the polynomials $\Delta_{j\nu}(x)$ are fixed, with determinant

$$J(x) \equiv |\Delta_{j\nu}|_{j,\,\nu=1,\ldots,k} \not\equiv 0$$

and let a system of relations of the form (90) hold (with these fixed $\Delta_{j\nu}$) for all members $(B_1(x;\alpha), \cdots, B_k(x;\alpha))$, $(P_1(x;\alpha), \cdots, P_k(x;\alpha))$ of some two sets of polynomials $\{B_1(x;\alpha), \cdots, B_k(x;\alpha)\}$, $\{P_1(x;\alpha), \cdots, P_k(x;\alpha)\}$, where each polynomial $B_j(x;\alpha)$, $P_j(x;\alpha)$ is of degree $\leq m$. Our immediate goal is to show, under the condition $J(x) \not\equiv 0$, that if the set $\{B(x;\alpha)\}$ is bounded, then the "solution" set $\{P(x;\alpha)\}$ is also bounded. For this purpose we introduce the cofactor $C_{\nu j}$ of the element $\Delta_{j\nu}$ in the matrix $||\Delta_{j\nu}||_{j,\,\nu=1,\ldots,k}$. Then by (90) we have

$$(91) \qquad P_\nu = \frac{\Sigma_{j=1}^k C_{\nu j} B_j}{J}$$

Since $J(x) \not\equiv 0$ and since $J(x)$ is obviously a polynomial, there exist a neighborhood (89) and a constant $\gamma > 0$ such that $|J(x)| \geq \gamma$ in (89). Hence, all elements B_j being bounded, there exists a constant M such that $|P_\nu(x)| \leq M$ in (89) for every $(P_1, \cdots, P_k)$ in $\{P(x;\alpha)\}$. This implies our desired conclusion.

Finally, we will prove the following more general result.

Lemma. *Let $\Delta_{j\nu}(t_1, \cdots, t_k)$ be a given set of k^2 polynomials, each of degree $\leq m$, with determinant*

$$J(t) \equiv |\Delta_{j\nu}|_{j,\,\nu=1,\ldots,k} \not\equiv 0$$

Let $\{L(\alpha)\}$ be a bounded set of linear transformations

$$(92) \qquad L(\alpha): \quad t_j(\alpha) = \Sigma_{\lambda=1}^k a_{j\lambda}(\alpha) x_\lambda, \qquad j = 1, \cdots, k$$

with determinants $\Delta_{L(\alpha)}$ *bounded away from zero. Let* $\{B_1(x; \alpha), \cdots, B_k(x; \alpha)\}$, $\{P_1(x; \alpha), \cdots, P_k(x; \alpha)\}$ *be two sets of polynomials in* $x_1, \cdots, x_k$, *each of degree* $\leq m$. *If a system of relations*

$$\Sigma_{\nu=1}^{k}\Delta_{j\nu}(t_1(\alpha), \cdots, t_k(\alpha))P_\nu(x; \alpha) = B_j(x, \alpha), \qquad j = 1, \cdots, k \tag{93}$$

viewed as relations in the x's, holds for each member $L(\alpha)$, $P_\nu(x; \alpha)$, $B_j(x; \alpha)$ *of the three sets, and if the set* $\{B(x; \alpha)\}$ *is bounded, then the set of "solutions"* $\{P(x; \alpha)\}$ *is also bounded.*

If the set $\{P(x; \alpha)\}$ were not bounded, then there would exist a sequence of systems L^s, P_ν^s, B_j^s, $s = 1, 2, \cdots$, each satisfying (93), such that the sum of the norms

$$\Sigma_{\nu=1}^{k}||P_\nu^s|| \tag{94}$$

is not bounded in s. By suitably thinning out the sequence we may assume that the transformations L^s are convergent.

Their limit transformation

$$L^0: \qquad t_j^0 = \Sigma_{\lambda=1}^{k}a_{j\lambda}^0 x_\lambda, \qquad j = 1, \cdots, k$$

is nonsingular. Therefore $J(t_1^0, \cdots, t_k^0)$, as a function in $x_1, \cdots, x_k$, is not identically zero. Hence there exists a neighborhood (89) such that $|J(t^0)| \geq 2\gamma > 0$ in (89). Obviously there exists an index s_0 such that for $s \geq s_0$, we also have $|J(t^s)| > \gamma$ in (89). Also, the cofactors $C_{\nu j}(t^s)$ are bounded in (89), and so are the functions $B_j^s(x)$. Hence, by (91), the polynomials P_ν^s are bounded, and thus the quantities (94) cannot be unbounded. This proves the lemma.

Now we can turn to the proof of Theorem 10. We consider the first half of the theorem. We take a generic element T of the set $\{T(\alpha)\}$ and write it in the form

$$T: \qquad x_j' = f_j(x) \equiv \Sigma_{s=1}^{\infty}P_s^j(x), \qquad j = 1, \cdots, k \tag{95}$$

Let $A(x_1, \cdots, x_k)$ be a fixed homogeneous polynomial of degree m, let $\Delta_\nu(x_1, \cdots, x_k)$ be the polynomial

$$\Delta_\nu(x_1, \cdots, x_k) = \frac{\partial A(x_1, \cdots, x_k)}{\partial x_\nu}$$

and let ρ be any fixed integer ≥ 1. Consider the series $A(f_1(x), \cdots, f_k(x))$. It is not hard to see that in the latter series the terms of degree $m + \rho$ consist of

$$\Sigma_{\nu=1}^{k}\Delta_\nu(P_1^1, \cdots, P_1^k)P_{\rho+1}^\nu + (\text{polynomial in } P_1^\lambda, \cdots, P_\rho^\lambda) \tag{96}$$

Hence if

$$g(x_1, \cdots, x_k) = \Sigma_{n=m}^{\infty} A_n(x)$$

then the terms of degree $m + \rho$ of the series $g(f_1(x), \cdots, f_k(x))$ are again of the form (96).

Now, if the transformation U of Theorem 10 has the form (86), and if we denote the partial derivative

$$\frac{\partial A_{n_j}^j(x_1, \cdots, x_k)}{\partial x_\nu}$$

by $\Delta_{j\nu}(x_1, \cdots, x_k)$, and if we put

$$t_j = P_1^j(x), \qquad j = 1, \cdots, k$$

then by (95) and the assumptions of Theorem 10, for each $j = 1, \cdots, k$ and each integer $\rho \geq 1$, an expression of the form

$$\Sigma_{\nu=1}^{k} \Delta_{j\nu}(t_1, \cdots, t_k) P_{\rho+1}^{\nu}(x) + (\text{polynomial in } P_1^\lambda, \cdots, P_\rho^\lambda) \tag{97}$$

is bounded for all $T = T(\alpha)$. We can now prove the boundedness of $P_s^\nu(x)$ for each $s = 1, 2, \cdots$, by induction on s. For $s = 1$, the boundedness is postulated in the theorem. Now suppose that we have already proved the boundedness of $P_1^\nu, \cdots, P_\rho^\nu$. Then the polynomial in $(P_1^\lambda, \cdots, P_\rho^\lambda)$ in the second term of (97) is bounded and hence the expressions

$$\Sigma_{\nu=1}^{k} \Delta_{j\nu}(t_1, \cdots, t_k) P_{\rho+1}^{\nu}(x_1, \cdots, x_k) \equiv B_j(x), \qquad j = 1, \cdots, k$$

are also bounded. It is now easy to see that the preceding lemma will apply, and we thus conclude that $P_{\rho+1}^\nu(x)$ is also bounded. This yields the first half of Theorem 10.

The second half follows from the first half by use of Theorem 6, since in this case, $\{T(\alpha)\}$ being a bounded group, we have $|\Delta_{T(\alpha)}| = 1$.

References

There seems to be no literature bearing on groups of transformations in terms of formal power-series, and the only references available are those bearing on analytic functions of several complex variables or on some other classes of "concrete" functions. As such they naturally pertain both to this chapter and to Chapter III. The main references are as follows:

1. H. Behnke, Über analytische Funktionen mehreren Veränderlichen. III. Abbildungen der Kreiskorper, *Hamburg Univ. Math. Sem. Abhandl.*, Vol. 7 (1930), pp. 329–341.

2. H. BEHNKE and E. PESCHL, Der Cartansche Eindeutigkeitssatz in unbeschränkten Körpern, *Math. Ann.*, Vol. 114 (1937), pp. 69–73.
3. H. BEHNKE and P. THULLEN, Theorie der Funktionen mehreren komplexer Veränderlichen, *Ergebnisse der Mathematik und ihrer Grenzgebiete,* Vol. 3, no. 3, Berlin, 1934.
4. S. BERGMAN, Über die Existenz von Repräsentantenbereichen, *Math. Ann.*, Vol. 102 (1929), pp. 430–446.
5. S. BOCHNER, Compact groups of differentiable transformations, *Annals of Math.*, Vol. 46 (1945), pp. 372–381.
6. C. CARATHÉODORY, Über die Abbildungen, die durch Systeme von analytische Funktionen von mehreren Veränderlichen erzeugt werden, *Math. Zeits.*, Vol. 34 (1932), pp. 758–792.
7. H. CARTAN, Les fonctions de deux variables complexes et le problème de la représentation analytique, *Jour. de Math. Pures et Appl.* (9), Vol. 10 (1931), pp. 1–114.
8. H. CARTAN, Sur les fonctions de plus variables complexes. L'itération des transforms intérieures d'un domaine borné, *Math. Zeits.* Vol. 35 (1932), pp. 760–773.
9. H. CARTAN, Sur les groupes de transformations analytiques, *Actualités Sci. Ind., Exposés Math. IX,* (1935) Paris, Hermann and Co.
10. E. PESCHL, Über den Cartan-Carathéodory Eindeutigkeitsätz, *Math. Ann.*, Vol. 119 (1943), pp. 131–139.
11. H. WELKE, Über die analytischen Abbildungen von Kreiskörpern und Hartogsschen Bereichen, *Math. Ann.*, Vol. 103 (1930), pp. 437–449.

The decisive references are No. 8 and No. 6, also No. 2. In No. 5 most of the results are generalized from complex analytic functions to ordinary differentiable functions, which however are assumed to be equiuniformly continuous and differentiable.

As for the group invariant measure, the so-called Haar measure, employed in the proof of Theorem 8, the most systematic account is:

12. ANDRE WEIL, L'intégration dans les groupes topologiques et ses applications, *Actualités Sci. Ind.*, no. 869, (1940) Paris, Hermann and Co.

The reader interested in the study of power-series whose coefficients are not necessarily numbers but elements of a general algebraic field, might be interested in examining for himself which of our theorems,

and to what extent, will remain valid. A very interesting problem arises in connection with Theorem 1. The circular group (50) is a very "small" group, indeed the only one parameter circular group. However, it "acts" in a manner which implicates all coordinates simultaneously, and the result is the very strong conclusion of Theorem 1. It would be worth investigating which modes of "acting" by groups with few parameters produce similar results, and what the statements would be for the case of real variables.

CHAPTER II

Basic Facts about Analytic Functions of Real and Complex Variables

It is the purpose of the present chapter to point out some basic similarities and dissimilarities between analytic functions of complex variables and analytic functions of real variables. In order to have a quick start we will define analyticity in terms of power-series. A function $f(z_1, \cdots, z_k)$ in a domain of its variables is *analytic* if in *some* neighborhood of every point $(z_1^0, \cdots, z_k^0)$ of the domain it is the sum of an (absolutely) convergent power series

(1) $$\Sigma_{n_1, \cdots, n_k=0}^{\infty} a_{n_1 \cdots n_k}(z_1 - z_1^0)^{n_1} \cdots (z_k - z_k^0)^{n_k}$$

The variables $z_1, \cdots, z_k$ may be all complex, all real, or some complex, some real. However, analytic functions in all types of variables will be reduced to those of complex variables, and hence we will begin with a discussion of the latter ones.

§1. Functions of Complex Variables

Writing $z_j = x_j + iy_j$, $j = 1, \cdots, k$, the space of k complex variables $z_1, \cdots, z_k$ is the ordinary Euclidean space E_{2k} of the $2k$ real variables $x_1, y_1, x_2, y_2 \cdots, x_k, y_k$. If $(z_1^0, \cdots, z_k^0)$ is any given point, a very convenient type of neighborhood will be the polycylinder

$$C(z^0, r): \qquad |z_j - z_j^0| < r_j, \qquad j = 1, \cdots, k$$

with $r_1 > 0, \cdots, r_k > 0$. If a power-series

(2) $$\Sigma_n a_{n_1 \cdots n_k} z_1^{n_1} \cdots z_k^{n_k}$$

is absolutely convergent for $|z_\nu| = R_\nu$, or, more generally, if

(3) $$|a_{n_1 \cdots n_k}| \leq \frac{M}{R_1^{n_1} \cdots R_k^{n_k}}$$

then inside the polycylinder

$$C(0, R): \quad |z_j| < R_j$$

the series (2) is majorized term-by-term by the series of

$$M\left(1-\frac{z_1}{R_1}\right)^{-1}\cdots\left(1-\frac{z_k}{R_k}\right)^{-1} \tag{4}$$

Therefore, if we arrange the terms of (2) into a simple series, that series will be uniformly convergent in $C(0, r)$ for any $r_j < R_j$. Since each monomial $z_1^{n_1}\cdots z_k^{n_k}$ is continuous in all variables, the sum-function $f(z_1, \cdots, z_k)$ is continuous in $C(0, R)$. Also, by a familiar theorem of Weierstrass on functions of one complex variable, since every monomial is analytic in each variable, the function f is likewise analytic in each variable, and the partial derivatives

$$\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_k} \tag{5}$$

can be obtained by formal differentiation of the series (2) in any simple ordering of its terms. Now, if (3) holds, the series for $\partial f/\partial z_1$ is majorized by the series

$$\frac{M}{R_1}\left(1-\frac{z_1}{R_1}\right)^{-2}\left(1-\frac{z_2}{R_2}\right)^{-1}\cdots\left(1-\frac{z_k}{R_k}\right)^{-1}$$

and similarly for $\partial f/\partial z_2, \cdots, \partial f/\partial z_k$. Thus the argument which has applied to $f(z)$, can be applied to each function (5), and by induction we find that $f(z)$ has mixed derivatives of all orders, and the derivatives can be obtained by formal differentiation of (2). In particular,

$$n_1!\cdots n_k!a_{n_1\cdots n_k} = \frac{\partial^{n_1+\cdots+n_k}f(0)}{\partial z_1^{n_1}\cdots\partial z_k^{n_k}} \tag{6}$$

Hence

Theorem 1. *An analytic function of complex variables is continuous and has (mixed) partial derivatives of all orders which are likewise analytic, and for the series* (1) *we have*

$$n_1!\cdots n_k!a_{n_1\cdots n_k} = \frac{\partial^{n_1+\cdots+n_k}f(z^0)}{\partial z_1^{n_1}\cdots\partial z_k^{n_k}} \tag{7}$$

If T_j is a domain in the z_j-plane, then the point set

$$[z_1 \subset T_1, \cdots, z_k \subset T_k] \tag{8}$$

is a domain in E_{2k}. It is called a (generalized) polycylinder, and we will also denote it by $(T_1, \cdots, T_k)$. If T_j' is contained with its boundary in T_j, and if its boundary is a smooth curve C_j, and if a function $f(z_1, \cdots, z_k)$ possesses the property in (8) that it is analytic in

each variable for all combinations of the other variables, then a repeated application of the ordinary Cauchy formula leads to the formula

$$(9) \quad f(z_1, \cdots, z_k) = \left(\frac{1}{2\pi i}\right)^k \int_{C_1} \frac{d\zeta_1}{\zeta_1 - z_1} \int_{C_2} \frac{d\zeta_2}{\zeta_2 - z_2} \cdots \int_{C_k} \frac{f(\zeta_1, \cdots, \zeta_k) d\zeta_k}{\zeta_k - z_k}$$

By a striking analysis, which we will reproduce in Chapter VII, F. Hartogs has proved that if $f(z)$ is analytic in each variable, it is continuous as a function in all variables.

For the present we will postulate explicitly that $f(z)$ is continuous in (8), but we do not make the assumption that f is analytic in $z_1, \cdots, z_k$ in (8). From the continuity it follows that f is uniformly continuous and bounded on the k-dimensional manifold

$$(10) \qquad [\zeta_1 \subset C_1, \cdots, \zeta_k \subset C_k]$$

and thus the repeated integral can be evaluated as a multiple integral. Now to show the analyticity of f in $z_1, \cdots, z_k$ in (8) it is only necessary to show that in a neighborhood of every point (z^0) of (8), f is representable by a multiple power-series of the form (1) convergent in that neighborhood. Without loss of generality we take the point (z^0) to be the origin $(z) = (0)$. If then each C_j is a circle $|\zeta_j| = R_j$, and $|z_j| < R_j$, z_j *fixed*, the expansion

$$\frac{1}{\zeta - z} = \Sigma_{n=0}^{\infty} \frac{z^n}{\zeta^{n+1}}, \qquad (|z| < |\zeta|)$$

leads to

$$(11) \qquad \frac{1}{(\zeta_1 - z_1) \cdots (\zeta_k - z_k)} = \Sigma_n \frac{z_1^{n_1} \cdots z_k^{n_k}}{\zeta_1^{n_1+1} \cdots \zeta_k^{n_k+1}}$$

the latter series being absolutely and uniformly convergent in (ζ) on $|\zeta_j| = R_j$ (for each fixed z_j in $|z_j| < R_j$). Since $f(\zeta_1, \cdots, \zeta_k)$ is bounded, we may substitute (11) into (9), and exchange the order of summation and integration. This leads to a series (2) with

$$(12) \qquad a_{n_1 \cdots n_k} = \frac{1}{(2\pi i)^k} \int_{C_1} \cdots \int_{C_k} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_k}{\zeta_1^{n_1+1} \cdots \zeta_k^{n_k+1}}$$

the series being convergent for $|z_j| < R_j$. For completeness we note that in the particular case under consideration in which C_j is the circle

$|\zeta_j| = R_j$, the integrals (9) and (12) are to be evaluated by putting

$$(13) \qquad \zeta_j = R_j e^{i\theta_j}, \qquad 0 \leq \theta < 2\pi, \qquad d\zeta_j = iR_j e^{i\theta_j} d\theta_j = i\zeta_j d\theta_j$$

Also if instead of the circles $|\zeta_j| = R_j$ we take some other circles $|\zeta_j| = R'_j \geq R_j$ (still lying together with their interiors in (8)), then the resulting power-series would have the same coefficients. This follows immediately from (7), and easily from (12). Thus we obtain

Theorem 2. *If a function $f(z_1, \cdots, z_k)$, all z_j complex, is continuous in a domain D, and if in the neighborhood of every point it is analytic in each variable, then $f(z)$ is analytic in D.*

Theorem 3. *If $f(z)$ is analytic in D, all z_j complex, then each expansion (1) is unique and is valid in every polycylinder $C(z^0, R)$, no matter how large, which is contained in D.*

As in the case of one complex variable, formula (9) leads very directly to important conclusions. The formula can be differentiated, that is

$$(14) \qquad \frac{\partial^{n_1+\cdots+n_k} f(z)}{\partial z_1^{n_1} \cdots \partial z_k^{n_k}} = \frac{n_1! \cdots n_k!}{(2\pi i)^k} \int_{C_1} \cdots \int_{C_k} \frac{f(\zeta)d\zeta_1 \cdots d\zeta_k}{(\zeta_1 - z_1)^{n_1+1} \cdots (\zeta_k - z_k)^{n_k+1}}$$

If a sequence of analytic functions in a common domain D is uniformly convergent in every compact set S of D then the limit function is again analytic, and the derivatives converge uniformly on S to the derivatives of the limit. In particular, the power-series around any point converge term-by-term. If (2) converges in $C(0, R)$, and (z^0) is a point in $C(0, R)$, then the series (1) can be obtained by formal manipulation. Furthermore, if the functions

$$(15) \qquad z_j = \varphi_j(\zeta_1, \cdots, \zeta_l), \qquad \varphi_j(0, \cdots, 0) = 0, \qquad j = 1, \cdots, k$$

are power series convergent in $|\zeta_\lambda| < \rho_\lambda$, and if their values lie in $C(0, R)$ then the function $f(\varphi_1(\zeta), \cdots, \varphi_k(\zeta))$ is again analytic in the same polycylinder $|\zeta_\lambda| < \rho_\lambda$, and its power-series can be obtained by formal substitution. Thus we obtain the theorem that an analytic function of analytic functions is again analytic.

§2. Functions of Real Variables

If $(z_1^0, \cdots, z_k^0)$, $z_j^0 = x_j^0 + iy_j^0$, is a point of E_{2k}, then a *real environment* of that point will be any point set containing a k-dimen-

sional rectangle

$$(16) \qquad |x_j - x_j^0| < R_j, \qquad y_j = y_j^0, \qquad j = 1, \cdots, k$$

Since the derivatives (7) can be formed by using only points of the environment, we see that if a power-series vanishes in a real environment of its center, it vanishes identically. Now let $f(z)$ be analytic in a domain D in E_{2k} and let $f(z)$ vanish in an environment (16) of a point $P(z^0)$ in D. First of all, it vanishes at all points of a corresponding polycylinder $C(z^0, R^0)$. If $P(z')$ is a point of that polycylinder, $f(z)$ vanishes in $C(z', R')$, and so forth. But any point of D can be connected with $P(z^0)$ by a finite chain of polycylinders, and thus $f(z)$ vanishes in D. Hence

Theorem 4. *If $f_1(z)$ is analytic in a domain D_1, and $f_2(z)$ in a domain D_2, if the intersection of D_1 and D_2 is a nonempty domain, and if $f_1(z)$, $f_2(z)$ have equal values in a real environment of a point of $D_1 \cdot D_2$, then $f_1(z)$ and $f_2(z)$ are analytic continuations of each other; i.e. there exists a unique function $f(z)$ analytic in $D_1 + D_2$ which coincides with f_1 in D_1 and with f_2 in D_2.*

More generally

Theorem 5. *If a domain D is the union of domains D_α, (α is an arbitrary index), if each intersection $D_\alpha \cdot D_\beta$ is either empty or a domain, if f_α is analytic in D_α, and if, whenever $D_\alpha \cdot D_\beta \neq 0$, f_α and f_β have equal values on some real environment, then there exists an analytic function $f(z)$ in D which coincides with f_α in D_α, for all α.*

Any domain T in the real variables $x_1, \cdots, x_k$ can be placed in the space of $z_j = x_j + iy_j$ by putting $y_j = 0$. If $f(x_1, \cdots, x_k)$ is analytic in T, if (x_j^α) is any point of T, and if

$$f(x) \equiv \Sigma_n a^\alpha_{n_1 \cdots n_k}(x_1 - x_1^\alpha)^{n_1} \cdots (x_k - x_k^\alpha)^{n_k}$$

in some rectangle $|x_j - x_j^\alpha| < R_j$, then we define D_α as the polycylinder $|z_j - x_j^\alpha| < R_j$, and $f_\alpha(z)$ as the function

$$\Sigma_n a^\alpha_{n_1 \cdots n_k}(z_1 - x_1^\alpha)^{n_1} \cdots (z_k - x_k^\alpha)^{n_k}$$

On the basis of Theorem 5 we now have:

Theorem 6. *If $f(x_1, \cdots, x_k)$ is analytic in T, all x_j real. then there exists in the space of the complex variables $z_j = x_j + iy_j$ a neighborhood D of T, and an analytic function $F(z_1, \cdots, z_k)$ in D, such that $F(x) \equiv f(x)$ in T.*

Also, if two neighborhoods D of T intersect in a domain, the corresponding functions $F(z)$ are identical in that domain by Theorem 4.

Thus $F(z)$ is a unique continuation of $f(x)$, and as such will be denoted simply by $f(z_1, \cdots, z_k)$. Furthermore, if $f_1(x)$ and $f_2(x)$ are analytic in the domains T_1 and T_2 respectively, and if T_1, T_2 intersect in a domain, then the complex neighborhoods D_1 and D_2 as constructed before will likewise intersect in a domain. Thus Theorems 4 and 5 also apply to functions of real variables.

However, we point out emphatically that the second part of Theorem 3 does not apply for real variables. It is not true that the expansion (1) which has been postulated to hold in a sufficiently small rectangle around the center will of necessity be valid in *every* rectangle in T. The size of the optimal rectangles will obviously depend on the thickness of the neighborhood D of T into which $f(x)$ can be continued. It is also not true that a uniform limit of analytic functions of real variables in a common domain T is again analytic, and if it happens to be analytic, it cannot be asserted that the power-series converge term-by-term. But the following result is true. Let $f(x_1, \cdots, x_k)$ be convergent in a k-dimensional rectangle about the origin as a center and let $\varphi_j(\xi_1, \cdots, \xi_l)$ be k power-series each convergent in a rectangle about $(\xi) = (0)$, and each vanishing at $(\xi) = (0)$. Then the series $f(\varphi_1(\xi), \cdots, \varphi_k(\xi))$ is again convergent in a rectangle about $(\xi) = (0)$, and this composite series can be obtained by ordinary manipulation. Hence it is true, even in real variables, that an analytic function of an analytic function is again analytic. This can be proved directly by the familiar method of majorization of all power-series occurring, or it can be reduced to the case of complex variables in the following way. Putting $z_j = x_j + iy_j$, we know that there exists a continuation $f(z_1, \cdots, z_k)$ in some complex neighborhood A of the origin, and again, putting $\zeta_\lambda = \xi_\lambda + i\eta_\lambda$, the functions $z_j = \varphi_j(\zeta_1, \cdots, \zeta_k)$ will likewise exist in a complex neighborhood of the origin; also, if the latter neighborhood is sufficiently small, the values $(\varphi_1, \cdots, \varphi_k)$ will lie in A, and thus we may apply the "complex" theorem.

§3. FUNCTIONS OF MIXED VARIABLES

If the series $f(z) = \Sigma a_n z^n$ converges for $|z| < R$, then the series $\Sigma a_n(x + iy)^n$, if rewritten as a double series

$$\Sigma_{p,\,q=0}^{\infty} a_{pq} x^p y^q$$

will converge absolutely for $|x| < \dfrac{R}{2}$, $|y| < \dfrac{R}{2}$, say. More generally, if some or all variables z_j in the series (2) are complex, and if we sub-

stitute $z_j = x_j + iy_j$, then we obtain a convergent power-series in all real variables occurring. Furthermore, if $f = u + iv$, then the real and imaginary parts u and v are each analytic in all real variables involved, and the Cauchy-Riemann equations give

$$\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial y_j} = 0, \qquad \frac{\partial u}{\partial y_j} + \frac{\partial v}{\partial x_j} = 0 \tag{17}$$

Now, assume that the function f, as function of its real variables, can be analytically continued to a larger domain. This continuation also continues the functions u, v, and hence the functions $\partial u/\partial x_j - \partial v/\partial y_j$, $\partial u/\partial y_j + \partial v/\partial x_j$. However, since the latter functions vanish in the original domain, they also vanish in the new domain. In other words, if an analytic function of real variables, $x_1, x_2, x_3, \cdots, x_n (n \geq 2)$ can be continued and if before continuation it involves the variables x_1, x_2 only in the combination $x_1 + ix_2$, i.e. if before continuation it is actually analytic in the variables $x_1 + ix_2, x_3, \cdots, x_n$ then this situation prevails after continuation.

We are pointing out this fact mainly because of the following very peculiar theorem some versions of which will be proved in Chapter IV. If $n \geq 3$, if $f(x_1, \cdots, x_n)$ is an analytic function of the real variables $x_1, \cdots, x_n$ on the boundary of a bounded n-dimensional domain, and if it involves x_1, x_2 only as $x_1 + ix_2$, then it can be continued analytically into the total interior of the domain. For $n = 2$ the theorem does not hold. A function $f(x + iy)$ may be analytic on the boundary of any domain, and not in its interior.

§4. CONJUGATE COMPLEX VARIABLES

Conjugate complex numbers will be denoted by bars. We start off with a lemma.

Theorem 7. *If*

$$f(z_1, \zeta_1, z_2, \zeta_2, \cdots, z_p, \zeta_p) \tag{18}$$

is a function, given as a power-series, of $2p$ complex variables in a $4p$-dimensional neighborhood of the origin $z_1 = \zeta_1 = z_2 = \zeta_2 = \cdots = z_p = \zeta_p = 0$, and if the function vanishes on the $2p$-dimensional manifold

$$\zeta_1 = \bar{z}_1, \qquad \zeta_2 = \bar{z}_2, \qquad \cdots, \qquad \zeta_p = \bar{z}_p \tag{19}$$

that is, if

$$f(z_1, \bar{z}_1, z_2, \bar{z}_2, \cdots, z_p, \bar{z}_p) = 0 \tag{20}$$

then (18) *vanishes identically, that is*

(21) $$f(z_1, \zeta_1, \cdots, z_p, \zeta_p) \equiv 0$$

Proof. We introduce the new *complex* variables

(22) $$x_j = \frac{z_j + \zeta_j}{2}, \qquad y_j = \frac{z_j - \zeta_j}{2i}, \qquad j = 1, \cdots, p$$

This is nonsingular linear transformation between the variables (z_j, ζ_j) and (x_j, y_j), and its inverse is

$$z_j = x_j + iy_j, \qquad \zeta_j = x_j - iy_j$$

In the new variables, the manifold (19) is the manifold

(23) $$x_j = \text{real}, \qquad y_j = \text{real}$$

Furthermore, in the new variables, the power-series (18) is a new power-series, again convergent, and if it vanishes on (23) it vanishes identically (by Theorem 4), and thus the original power-series vanishes identically.

Now, if $f(x, y)$ is an ordinary function (not necessarily analytic), with continuous derivatives $\partial f/\partial x$, $\partial f/\partial y$, in the *real* variables x, y, we may introduce purely symbolically, the new "variables"

(24) $$z = x + iy, \qquad \bar{z} = x - iy$$

with

$$x = \frac{z + \bar{z}}{2}, \qquad y = \frac{z - \bar{z}}{2i}$$

and write, always symbolically,

(25) $$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right)$$

Formulas (25) are arrived at by ordinary calculus as if the letter i in (24) were a real number. Putting $f = u + iv$ we have

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) \equiv \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right\}$$

Thus the Cauchy-Riemann equations, see (17), are "equivalent" with

$$\frac{\partial f}{\partial \bar{z}} = 0$$

More generally, if

$$f(x_1, y_1, \cdots, x_k, y_k) \tag{26}$$

has continuous partial derivatives of the first order, then it is an analytic function of the complex variables $z_j = x_j + iy_j$ if and only if, symbolically,

$$\frac{\partial f}{\partial \bar{z}_1} = 0, \quad \cdots, \quad \frac{\partial f}{\partial \bar{z}_k} = 0 \tag{27}$$

It is worth pointing out that if (26) is known to be analytic, relations (27) may be taken literally. In fact we continue (26) into a complex neighborhood of x_j, y_j, and then the complex substitution (22) leads to the literal relations

$$\frac{\partial f}{\partial \zeta_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - \frac{1}{i}\frac{\partial f}{\partial y_j}\right) \tag{28}$$

Now, the literal meaning of the relations (27) is that the function (28) vanishes on the manifold (19). However, by Theorem 7, this implies that the functions (28) vanish identically, that is, the power-series of (26), if transcribed into the "independent" variables $z_1, \bar{z}_1, \cdots, z_k, \bar{z}_k$, shall contain only monomials $z_1^{n_1} \cdots z_k^{n_k}$.

If $f_1, \cdots, f_n$ are functions in the real variables $x_1, \cdots, x_n$, with continuous first derivatives, and if we make (cogredient) non-singular linear transformations with constant coefficients

$$\begin{aligned} f_m &= \Sigma_{\mu=1}^{n} c_{m\mu}\varphi_\mu, \\ x_m &= \Sigma_{\mu=1}^{n} c_{m\mu}\xi_\mu, \end{aligned} \qquad m = 1, \cdots, n$$

then the Jacobian remains unchanged,

$$\frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)} \equiv \frac{\partial(\varphi_1, \cdots, \varphi_r)}{\partial(\xi_1, \cdots, \xi_n)}$$

This algebraic identity prevails for symbolic transformations. Thus, if we have real functions

$$u_j = u_j(x_1, y_1, \cdots, x_k, y_k), \qquad v_j = v_j(x_1, y_1, \cdots, x_k, y_k)$$

$j = 1, \cdots, k$, and if we introduce the symbols $f_j = u_j + iv_j$, $\bar{f}_j = u_j - iv_j$, $z_j = x_j + iy_j$, $\bar{z}_j = x_j - iy_j$, we obtain

$$\frac{\partial(u_1, v_1, \cdots, u_k, v_k)}{\partial(x_1, y_1, \cdots, x_k, y_k)} \equiv \frac{\partial(f_1, \bar{f}_1, \cdots, f_k, \bar{f}_k)}{\partial(z_1, \bar{z}_1, \cdots, z_k, \bar{z}_k}$$

A permutation of variables is a linear transformation, hence this is also equal to

$$\frac{\partial(f_1, f_2, \cdots, f_k; \bar{f}_1, \bar{f}_2, \cdots, \bar{f}_k)}{\partial(z_1, z_2, \cdots, z_k; \bar{z}_1, \bar{z}_2, \cdots, \bar{z}_k)} \tag{29}$$

Now, if $f_j = f_j(z_1, \cdots, z_k)$, i.e. if the f_j are analytic in the z's, then $\partial f_j/\partial \bar{z}_m = 0$ and $\partial \bar{f}_j/\partial z_m = 0$ and by direct evaluation of (29), it is the product

$$\frac{\partial(f_1, \cdots, f_k)}{\partial(z_1, \cdots, z_k)} \cdot \frac{\partial(\bar{f}_1, \cdots, \bar{f}_k)}{\partial(\bar{z}_1, \cdots, \bar{z}_k)}$$

Hence

Theorem 8. *If $f_j(z_1, \cdots, z_k)$, $j = 1, \cdots, k$, are analytic functions of complex variables, and if $f_j = u_j + iv_j$, then*

$$\frac{\partial(u_1, v_1, \cdots, u_k, v_k)}{\partial(x_1, y_1, \cdots, x_k, y_k)} = \left|\frac{\partial(f_1, \cdots, f_k)}{\partial(z_1, \cdots, z_k)}\right|^2$$

We can now prove an important result on implicit functions for analytic functions of real or complex variables.

Theorem 9. *If the functions*

$$F_j(w_1, \cdots, w_k; z_1, \cdots, z_l), \qquad j = 1, \cdots, k \tag{30}$$

are analytic functions of $k + l$ variables in the neighborhood of the origin, if $F_j(0; 0) = 0$, and if

$$\frac{\partial(F_1, \cdots, F_k)}{\partial(w_1, \cdots, w_k)} \neq 0 \qquad \text{for} \qquad w = z = 0 \tag{31}$$

then the equations

$$F_j(w_1, \cdots, w_k; z_1, \cdots, z_l) = 0, \qquad j = 1, \cdots, k \tag{32}$$

have a unique solution

$$w_j = w_j(z_1, \cdots, z_l) \tag{33}$$

vanishing for $(z) = (0)$ and analytic in the neighborhood of the origin.

Proof. We will reduce this theorem to the corresponding theorem in real variables for differentiable functions. We first assume that all our variables are complex. Writing the system (32) as a system of $2k$ real equations with the unknown real functions u_j, v_j, we see by Theorem 8 that its Jacobian is again $\neq 0$ at $(w) = (z) = (0)$. Hence it has solutions with continuous partial derivatives of first order

unique in so far as they vanish at the origin. The reader will verify that the chain rule for differentiation prevails for our symbolic quantities. Therefore, differentiating each relation (32) with respect to $\bar{z}_m$ we obtain

$$\Sigma_{p=1}^{k} \frac{\partial F_j}{\partial w_p} \cdot \frac{\partial w_p}{\partial \bar{z}_m} + \Sigma_{\lambda=1}^{l} \frac{\partial F_j}{\partial z_\lambda} \cdot \frac{\partial z_\lambda}{\partial \bar{z}_m} = 0$$

However, z_λ and $\bar{z}_m$ are "independent," and hence

$$\Sigma_{p=1}^{k} \frac{\partial F_j}{\partial w_p} \cdot \frac{\partial w_p}{\partial \bar{z}_m} = 0, \qquad j, m = 1, \cdot \cdot \cdot , k$$

Therefore, by ordinary linear algebra, (31) implies $\partial w_p/\partial \bar{z}_m = 0$ in a neighborhood of $(z) = (0)$ and thus our functions w_j are analytic in $z_1, \cdot \cdot \cdot , z_l$.

If the $k + l$ variables were real to start with, we can continue our functions (30) into a complex neighborhood, and thus reduce the case of real variables to the case of complex variables.

There are many direct proofs of Theorem 9. Some proofs of the ordinary theorem can be carried over directly to the analytic case, or, Cauchy's method of undetermined coefficients with majorization of the power-series involved can be applied, or, finally, as in the simple case $F(w, z) = 0$, the Weierstrass-Hurwitz method of residues can be used.

§5. MAJORIZED FAMILIES OF FUNCTIONS

A family $\{f_\alpha(z)\}$ of analytic functions of real or complex variables in a common domain D will be called *majorized,* if corresponding to any point $(z_1^0, \cdot \cdot \cdot , z_k^0)$ of D the coefficients of the power-series (1) at that point satisfy an inequality

$$|a^\alpha_{n_1 \cdots n_k}| \leq A_{n_1 \cdots n_k}$$

with

$$A_{n_1 \cdots n_k} \leq \frac{M}{R_1^{n_1} \cdot \cdot \cdot R_k^{n_k}}$$

for some $R_j > 0$. This implies

$$\Sigma A_{n_1 \cdots n_k} r_1^{n_1} \cdot \cdot \cdot r_k^{n_k} < \infty$$

for $r_j < R_j$.

It is easy to see that derivatives of any fixed order of $f_\alpha(z)$ are again majorized, and that the functions $f_\alpha(z)$, and hence their derivatives of any order, are uniformly continuous and uniformly bounded on any

compact point set S of D. In the case of complex variables, and only then, will, conversely, uniform boundedness imply majorization, see (9) and (12).

As in the case of one complex variable, if follows for several complex variables, that any majorized sequence contains a sub-sequence which with all its derivatives is uniformly convergent on every compact set S in D, and that any majorized sequence will be so convergent, if it is convergent in a neighborhood of a point, or only in a real environment of a point, or if its power-series for one point are convergent term-by-term. These assertions carry over to majorized families in *real* variables, on the basis of the following statement which the reader will easily verify. If the family $f_\alpha(x_1, \cdots, x_k)$ is a majorized family on a real domain T, then there exists a *common* neighborhood D of T in the space $z_j = x_j + iy_j$ on which all $f_\alpha(z_1, \cdots, z_k)$ exist and are again majorized.

If $f(z_1, \cdots, z_k; \zeta_1, \cdots, \zeta_l)$ is analytic in $k + l$ complex variables in the topological product of a domain $A(z)$ with a domain $B(\zeta)$, then the family $f_\zeta(z) \equiv f(z; \zeta)$, with the parameters ζ_λ varying over a compact set S_B in $B(\zeta)$, is majorized in $A(z)$. This is quite obvious, since, being analytic in (z, ζ), the function $f_\zeta(z)$ is bounded in the topological product of S_B with any compact set S_A in $A(z)$. However, the theorem also prevails for all variables (z, ζ) being real, as a continuation into complex neighborhoods of (z, ζ) will make manifest.

§6. A Theorem of E. E. Levi

If $K(x_1, \cdots, x_k; t_1, \cdots, t_l) \equiv K_t(x)$ is a majorized family in the variables $x_1, \cdots, x_k$, and if the coefficients of its power-series are each (bounded and) integrable in the parameters t_λ, and if $\varphi(t_1, \cdots, t_l)$ is integrable, but not necessarily bounded, over the point set T of the parameters, then the integral

$$F(x) = \int_T K(x; t)\varphi(t)dt_1 \cdots dt_l \tag{34}$$

is an analytic function. This follows by substituting power series for $K(x; t)$ and exchanging summation with integration, say. This type of reasoning for the analyticity breaks down if $K(x; t)$ has singularities and more sophisticated methods must be devised. One such method has been found by E. E. Levi. Since it has an important place in the theory of partial differential equations and can be readily expounded, we will exemplify it on a simple case.

Let $\varphi(t_1, t_2, t_3)$ be an analytic function of real variables in an open

neighborhood of the unit sphere

$$(35) \quad T: \qquad t_1^2 + t_2^2 + t_3^2 \leq 1$$

We will show that for $0 < \lambda < \frac{3}{2}$, the integral

$$(36) \quad F(x_1, x_2, x_3) = \iiint_T \frac{\varphi(t_1, t_2, t_3)dt_1dt_2dt_3}{[(x_1 - t_1)^2 + (x_2 - t_2)^2 + (x_3 - t_3)^2]^\lambda}$$

is analytic for

$$(37) \qquad x_1^2 + x_2^2 + x_3^2 < 1$$

The surface

$$(38) \qquad \xi_1^2 + \xi_2^2 + \xi_3^2 = 1$$

can be parametrized by

$$(39) \qquad \xi_1 = \sin \alpha \cos \beta, \qquad \xi_2 = \sin \alpha \sin \beta, \qquad \xi_3 = \cos \alpha$$

Introducing the third variable ρ, the reader will have no great difficulty in verifying that for each fixed x_1, x_2, x_3 in (37) the transformation

$$t_\nu - x_\nu = \rho \cdot (\xi_\nu - x_\nu) \qquad \nu = 1, 2, 3$$

that is

$$(40) \qquad t_\nu = x_\nu + \rho \cdot (\xi_\nu - x_\nu)$$

transforms (35) into

$$(41) \qquad 0 \leq \rho \leq 1, \qquad 0 \leq \alpha \leq \pi, \qquad 0 \leq \beta \leq 2\pi$$

Also

$$\frac{\partial(t_1, t_2, t_3)}{\partial(\rho, \alpha, \beta)} = \rho^2 D(\alpha, \beta; x)$$

where

$$D(\alpha, \beta; x) = \begin{vmatrix} \xi_1 - x_1 & \xi_2 - x_2 & \xi_3 - x_3 \\ \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \alpha \sin \beta & \sin \alpha \cos \beta & 0 \end{vmatrix}$$

Thus if we put

$$(42) \quad \Phi(\rho, t, x) \equiv \varphi(x_1 + \rho(t_1 - x_1), x_2 + \rho(t_2 - x_2), x_3 + \rho(t_3 - x_3))$$

the integral (36) will be

$$(43) \qquad \int_0^\pi d\alpha \int_0^{2\pi} d\beta \int_0^1 \frac{\rho^2 d\rho}{\rho^{2\lambda}} \frac{\Phi(\rho, \xi, x) \cdot D(\alpha, \beta; x)}{[\psi(x, \xi)]^\lambda}$$

where

$$\psi(x, t) = (x_1 - t_1)^2 + (x_2 - t_2)^2 + (x_3 - t_3)^2 \tag{44}$$

The reader will again have no great difficulty in verifying that if $\varphi(t_1, t_2, t_3)$ is analytic in (35), the function $\Phi(\rho, t, x)$ will be analytic in all seven variables, in an open neighborhood of

$$x_1^2 + x_2^2 + x_3^2 < 1, \qquad t_1^2 + t_2^2 + t_3^2 \leq 1 \qquad 0 \leq \rho \leq 1$$

Therefore, $\Phi(\rho, \xi, x)$ in (43) is a majorized family. Similarly $D(\alpha, \beta; x)$ is a majorized family. Finally $[\psi(x, \xi)]^{-\lambda}$ is a majorized family for any $\lambda \gtrless 0$ for x in (37) and ξ in a neighborhood of (38). In fact, for any $\epsilon > 0$, $\psi(x, t)$ is $\geq \epsilon^2$, if $x_1^2 + x_2^2 + x_3^2 \leq (1 - 2\epsilon)^2$ and $(1 - \epsilon)^2 < t_1^2 + t_2^2 + t_3^2 < (1 + \epsilon)^2$, and thus for the same values of (x, t), $[\psi(x, t)]^{-\lambda} \equiv \exp\{-\lambda \log \psi(x, t)\}$ is also analytic. Lastly if $\lambda < \frac{3}{2}$, then $\rho^{2-2\lambda}$ is integrable in $0 < \rho < 1$, and thus the analyticity of (43) follows from the original unsophisticated version of our theorem.

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These are only some of the principal references. No. 3 is different from the others, referring as it does to the specialized topic discussed in section 6. It has become an indispensable tool in the theory of partial equations of elliptic type. As a typical reference we quote:

9. G. GIRAUD, Sur le problème de Dirichlet généralisé, équations non linéaires à m variables, *Paris, École Norm. Sup., Ann. sci.* Ser. 3, Vol. 43 (1926), pp. 1–128.

The same author had subsequently many more papers, mostly published in French periodicals. See also:

10. E. HOPF, Ueber den funktionalen, insbesondere den analytischen Character der Lösungen elliptischer Differentialgleichungen zweiter Ordnung, *Math. Zeits.*, Vol. 34 (1931), pp. 193–233, esp. p. 224.

For an application of Levi's procedure to a problem other than differential equations see:

11. S. BOCHNER, Completely monotone functions of the Laplace operator for torus and sphere, *Duke Math. J.*, Vol. 3 (1937), pp. 488–502, esp. p. 495.

In this connection mention should be made of the paper in which a method other than Levi's, but rather restricted in scope is introduced:

12. H. LEWY, Über den analytischen character der Lösungen elliptischer Differentialgleichungen, *Gesellsschaft d. wissens., Göttingen, Math. Phys. Kl. Nachr.*, 1927 pp. 178–184.

CHAPTER III

Analytic Mappings with a Fixed Point

§1. Analytic Homeomorphism of the Entire Space into a Part of Itself

An entire function of one complex variable cannot omit the neighborhood of a value (Weierstrass) and, as a matter of fact, it cannot omit more than one value (Picard). A generalization to several variables would be that if $f(x, y)$, $g(x, y)$ are analytic functions for all values of the complex variables x, y, and if they are independent, that is, if their Jacobian does not vanish identically, then the point set $u = f(x, y)$, $v = g(x, y)$ cannot omit a neighborhood of a point in the (complex) (u, v)-space. However, such a generalization fails utterly of holding (Fatou, Bieberbach).

Theorem 1. *There exists a pair of entire functions*

$$(1) \qquad \begin{aligned} u &= x + \text{(higher powers)} \\ v &= y + \text{(higher powers)} \end{aligned}$$

which represent a topological mapping of the space of complex variables (x, y) *into a proper part of itself. The complement of the image contains an open set and*

$$(2) \qquad \frac{\partial(u, v)}{\partial(x, y)} \equiv 1$$

for all x, y.

Proof. We start off with the functional equation

$$(3) \qquad \varphi(4x, 4y) - 4\varphi(x, y) = a\varphi(2x, -2y)^2 + b\varphi(2x, -2y)^5$$

(a, b are constants). We want to show that (3) has a unique solution of the form

$$(4) \qquad \varphi(x, y) = x + y + \Sigma_{p+q \geq 2} A_{pq} x^p y^q$$

and that the series converges in some neighborhood of the origin. On substituting (4) in (3), we obtain

$$(5) \qquad \Sigma_{p+q \geq 2} (4^{p+q} - 4) A_{pq} x^p y^q = \Sigma_{p+q \geq 2} \rho_{pq} x^p y^q$$

where

$$\rho_{pq} \equiv \rho_{pq}(a, b, A_{\mu\nu})$$

is a polynomial in a, b and those $A_{\mu\nu}$ for which $\mu + \nu < p + q$. Thus the relation

$$A_{pq} = \frac{\rho_{pq}(a, b, A_{\mu\nu})}{4^{p+q} - 4}$$

will inductively determine a unique set of coefficients. In order to prove the convergence of (4) we compare (3) with

$$\psi(2x, 2y) = a\{2(x + y) + \psi(2x, 2y)\}^2 + b\{2(x + y) + \psi(2x, 2y)\}^5 \tag{6}$$

of which we set up a solution

$$\psi(x, y) = \Sigma_{p+q \geq 2} B_{pq} x^p y^q \tag{7}$$

We obtain a similar recurrence formula

$$B_{pq} = \frac{\sigma_{pq}(a, b, B_{\mu\nu})}{2^{p+q}}$$

with $\mu + \nu < p + q$ again holding. Now, the reader will verify that for any a, b, $C_{\mu\nu}$ the inequality

$$|\rho_{pq}(a, b, C_{\mu\nu})| \leq \sigma_{pq}(|a|, |b|, |C_{\mu\nu}|)$$

holds. Also $4^{p+q} - 4 \geq 2^{p+q}$ for $p + q \geq 2$. Hence it is sufficient to prove that (7) is convergent. Finally we compare (6) with

$$\Psi(t) = a\{t + \Psi(t)\}^2 + b\{t + \Psi(t)\}^5$$

It has a unique formal solution

$$\Psi(t) = \Sigma_{n=2}^{\infty} \alpha_n t^n \tag{8}$$

and since $\Psi(2(x + y))$ is a formal solution of (6) we have only to verify convergence of (8). Now, $u \equiv \Psi(t)$ is a formal solution, which vanishes at the origin, of the equation $F(t, u) = 0$, where $F(t, u) = u - a(t + u)^2 - b(t + u)^5$. Since $\partial F/\partial u = 1$ for $t = u = 0$, we know by Theorem 9, Chapter II, that this equation does have an analytic solution and therefore (8) must be it.

Now, let R, $0 < R \leq \infty$, be the largest number such that (4) converges in the polycylinder $|x| < R$, $|y| < R$. We can write (3) in the form

$$\varphi(x, y) = 4\varphi\left(\frac{x}{4}, \frac{y}{4}\right) + a\varphi\left(\frac{x}{2}, -\frac{y}{2}\right)^2 + b\varphi\left(\frac{x}{2}, -\frac{y}{2}\right)^5 \tag{9}$$

and by analytic continuation from the neighborhood of the origin, we see that $\varphi(x, y)$ satisfies the last relation in the whole polycylinder.

Now if R were finite, we could use relation (9) to find an analytic continuation into the polycylinder with $R' = 2R$. Thus $R = \infty$, and $\varphi(x, y)$ is an entire function.

We construct $\varphi(x, y)$ for the special values $a = -5$, $b = 2$. Putting

$$T: \quad \begin{aligned} f(x, y) &= \varphi(x, y) = x + y + (\text{higher powers}) \\ g(x, y) &= \varphi(2x, -2y) = 2(x - y) + (\text{higher powers}) \end{aligned}$$

we obtain

$$\begin{aligned} f(2x, -2y) &= g(x, y) \\ g(2x, -2y) &= 4f(x, y) + 2g(x, y)^5 - 5g(x, y)^2 \end{aligned}$$

If we define

$$\text{(10)} \qquad \begin{aligned} u(x, y) &\equiv f(2x, -2y) \\ v(x, y) &\equiv g(2x, -2y) \end{aligned}$$

then we find

$$\text{(11)} \qquad \begin{aligned} g(x, y) &= u(x, y) \\ f(x, y) &= \tfrac{1}{4}\{v - 2u^5 + 5u^2\} \end{aligned}$$

Putting

$$J(x, y) = \frac{\partial(f(x, y), g(x, y))}{\partial(x, y)}$$

we have on the one hand

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(f(2x, -2y), g(2x, -2y))}{\partial(x, y)} = -4J(2x, -2y)$$

and on the other hand

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \\ 4\dfrac{\partial f}{\partial x} + (10g^4 - 10g)\dfrac{\partial g}{\partial x} & 4\dfrac{\partial f}{\partial y} + (10g^4 - 10g)\dfrac{\partial g}{\partial y} \end{vmatrix}$$
$$= -4J(x, y)$$

By comparison, and subsequent iteration, we have

$$J(x, y) = J\left(\frac{x}{2^n}, \frac{y}{(-2)^n}\right) = \lim_{n\to\infty} J\left(\frac{x}{2^n}, \frac{y}{(-2)^n}\right)$$
$$= J(0, 0) = -4$$

Thus $J(x, y) =$ constant. Furthermore it follows from (10) that if $(f(x, y), g(x, y))$ should map two different points (x_1, y_1), (x_2, y_2) into

equal points, it would map $(x_1/2^n, y_1/(-2)^n)$, $(x_2/2^n, y_2/(-2)^n)$ into equal points. Thus in every neighborhood of the origin there would be different points which are mapped into equal points. Since $J(0, 0) \neq 0$ this is contradictory (by the corresponding theorem on ordinary differentiable functions). Thus T is a topological transformation of the space into part of itself. We are going to show that the image avoids a neighborhood of the point $x = 1$, $y = 1$. If for some x, y

$$u(x, y) = 1 + \rho, \qquad v(x, y) = 1 + \sigma \qquad \text{with} \qquad |\rho| \leq \epsilon, |\sigma| \leq \epsilon$$

then by (11)

$$\begin{aligned} g &= 1 + \rho \\ f &= 1 + \tfrac{1}{4}(\sigma - 15\rho^2 - 20\rho^3 - 10\rho^4 - 2\rho^5) \end{aligned}$$

and thus for some $\epsilon < 1$, independent of (x, y), we have $|g - 1| \leq \epsilon$, $|f - 1| \leq \epsilon$ holding for the same ϵ as above. Thus by (10) the inequalities $|u(x, y) - 1| \leq \epsilon$, $|v(x, y) - 1| \leq \epsilon$ would imply the same pair of inequalities for $(x/2^n, y/(-2)^n)$. But $u(0, 0) = v(0, 0) = 0$, and we again have a contradiction.

Finally we obtain the function (1) of the theorem by changing $x + y$, $2(x - y)$ into x, y in $f(x, y)$, $g(x, y)$. This linear transformation is a one-to-one map of (x, y)-space into itself and preserves the properties of $f(x, y)$, $g(x, y)$ accordingly.

§2. THE UNIQUENESS THEOREM OF H. CARTAN

Following Carathéodory we will call a sequence of continuous functions in a domain D, or more generally, a sequence of continuous transformations of a domain D, *continuously convergent* in D, if the sequence is uniformly convergent in every compact set S of D. The intrinsic significance of the concept need not be discussed for the present.

Let D be a domain containing the origin, and consider transformations of D (by means of analytic functions) which leave the origin unchanged. Thus a transformation of this sort has the form

$$T: \qquad z'_j = f_j(z_1, \cdots, z_k) \qquad j = 1, \cdots, k$$

where the $f_j(z)$ are analytic in D and vanishing at the origin. Often we shall write T in the form

$$(12) \quad T: \quad z'_j = f_j(z) \equiv a_{j1}z_1 + \cdots + a_{jk}z_k + (\text{higher powers})$$

by which we mean that T is representable throughout D by means of analytic functions $f_j(z)$, and that in a neighborhood of the origin these functions have the expansions indicated. Now if a family of such transformations is majorized according to the definition in section 5, Chapter II, then their expansions, if looked upon as formal power-series, are weakly bounded according to the definition in section 3, Chapter I. Furthermore, *if a sequence of (analytic) transformations is bounded in a domain D, then the sequence is continuously convergent if and only if it is weakly convergent.* The fact that a continuously convergent bounded sequence is weakly convergent follows immediately from formula (12), Chapter II. The fact that a weakly convergent bounded sequence in D is continuously convergent in a neighborhood of the origin follows immediately from the power-series development of the functions. We can repeat this argument with any point of this neighborhood; and so, by the method used in analytic continuation, extend the domain of continuous convergence throughout D. This equivalence of continuous convergence and weak convergence for bounded sequences of analytic transformations leads to many concrete interpretations and conclusions from the results of Chapter I. We first state the following conclusion from Theorems 3 and 9 of Chapter I.

Theorem 2. *If D is a domain in the space of k complex variables containing the origin, and if*

$$z_j' = f_j(z) = z_j + (\text{higher powers}) \qquad j = 1, \cdots, k \tag{13}$$

is a map of D into part of itself, the $f_j(z)$ being analytic in D, then

$$f_j(z) \equiv z_j, \qquad j = 1, \cdots, k$$

whenever D is bounded. Instead of assuming that D is bounded it is sufficient to assume that there exist k power-series

$$w_j(z) = \Sigma_{n=n_j}^{\infty} A_n^j(z) \qquad (n_j \geq 1)$$

with the following three properties: (i) *each series is convergent in a neighborhood of the origin and can be continued along any path in D;* (ii) *the values of the functions thus continued are bounded and* (iii)

$$\frac{\partial(A_{n_1}^1, \cdots, A_{n_k}^k)}{\partial(z_1, \cdots, z_k)} \not\equiv 0$$

Remark. As in section 1 of Chapter I, $A_n^j(z)$ is a homogeneous polynomial in $z_1, \cdots, z_k$ of degree n.

We emphasize that the functions $w_j(z)$ need not be (one-valued) analytic functions on D; we only require that each $w_j(z)$ shall exist on the universal covering domain of D.

By combining Theorems 1 and 2 of the present chapter we can obtain an interesting result. Take the functions $u(x, y)$, $v(x, y)$ of Theorem 1. (Remember that x and y are here complex variables.) These two functions map the total (x, y)-space topologically into a domain D which omits an open set. Hence they are a pair of functions beginning with x, y respectively which map the domain D into part of itself without being the identity. Hence D does not fall under Theorem 2. Let us denote by (x_0, y_0) a point which together with a closed sphere $|x - x_0|^2 + |y - y_0|^2 \leq \epsilon^2$ lies outside of D. By Theorem 1 there exists such a sphere with $\epsilon > 0$. Then D is contained in the domain

$$|x - x_0|^2 + |y - y_0|^2 > \epsilon^2$$

and thus by Theorem 2 there *cannot* exist a pair of functions $w_1(x, y)$, $w_2(x, y)$ in the domain $|x - x_0|^2 + |y - y_0|^2 > \epsilon^2$ which are bounded and have a non-vanishing Jacobian. On writing

$$x' = \frac{x - x_0}{\epsilon}, \qquad y' = \frac{y - y_0}{\epsilon}$$

we conclude the same result for the exterior of the unit sphere $|x|^2 + |y|^2 > 1$, that is, there does *not* exist a pair of functions $w_1(x, y)$, $w_2(x, y)$ in the domain $|x|^2 + |y|^2 > 1$ which are bounded and have a non-vanishing Jacobian.

§3. CRITERION FOR A TRANSFORMATION WITH A FIXED POINT TO BE AN AUTOMORPHISM

Transformation (13) is a very special kind among those transformations of type (12) for which

$$|\Delta_T| = 1 \tag{14}$$

where, as in Chapter I, Δ_T is the value of the Jacobian of (12) at the origin.

Now, H. Cartan and Carathéodory have found that the mere assumption (14) leads to a significant conclusion. All other assumptions in Theorem 2 being unchanged, relation (14) can hold for a transformation (12) of a bounded domain into part of itself only if our transformation is an automorphism, that is, a topological transformation of D into (all of) itself. We will have a simpler proof of

the theorem, again generalizing it to majorized semigroups of real transformations.

Consider a real analytic transformation

$$T: \qquad u_j = a_1^j x_1 + \cdots + a_k^j x_k + g_j(x), \qquad j = 1, \cdots, k$$

The series $g_j(x)$ are terms of order ≥ 2, also $\Delta_T \neq 0$, and T is a one-to-one transformation of a domain D_x containing $(x) = (0)$ into a domain D_u containing $(u) = (0)$. The quadratic form

$$v_1^2 + \cdots + v_k^2 = \Sigma_{j=1}^k (a_1^j x_1 + \cdots + a_k^j x_k)^2$$

is strictly positive definite. Hence there exists a number $A > 0$ such that

$$v_1^2 + \cdots + v_k^2 \geq A^2 \cdot (x_1^2 + \cdots + x_k^2)$$

On the other hand, in some sphere $x_1^2 + \cdots + x_k^2 \leq \sigma^2$, $g_1(x)^2 + \cdots + g_k(x)^2 \leq B^2 \cdot (x_1^2 + \cdots + x_k^2)^2$. Hence

$$(u_1^2 + \cdots + u_k^2)^{\frac{1}{2}} \geq [A - B \cdot (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}] \cdot (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$$

Thus, putting $\rho = \min(\sigma, A/2B)$, $r_0 = A\rho/2$, we see that on the boundary of

$$x_1^2 + \cdots + x_k^2 < \rho^2 \tag{15}$$

we have $u_1^2 + \cdots + u_k^2 \geq r_0^2$. Since T is one-to-one, the image of (15) is a domain. This domain contains the origin, and its boundary lies outside the open sphere of radius r_0. Therefore, D_u contains the sphere

$$u_1^2 + \cdots + u_k^2 < r_0^2 \tag{16}$$

Now consider a family of one-to-one analytic transformations $\{T\}$, with the origin as fixed point and assume that they are a majorized family on a common domain D_x and that $|\Delta_T| \geq c > 0$. It is not hard to see that in this case the quantities A, σ, B, ρ, r_0 can be chosen in common, and thus we obtain the following lemma:

Lemma 1. *The images D_u contain a common sphere* (16).

Our conclusion from this lemma will concern any majorized sequence of transformations with no fixed point requirement. If T, T_n, $\cdots$, etc. is the transformation of a domain D, then the image of a point P, or of any subset S of D will be denoted by $T(P)$, $T_n(P)$, $T(S)$, $T_n(S)$, etc.

Theorem 3. *If*

$$T_n: \qquad u_j = f_j^n(x_1, \cdots, x_k), \qquad j = 1, \cdots, k$$

$n = 1, 2, 3, \cdots$, *is a majorized sequence of transformations of a domain D in E_k (by means of analytic functions $f_j^n(x)$), if the sequence is continuously convergent to a transformation*

$$T_0: \qquad u_j = f_j^0(x)$$

if P is any point of D, if $P_0 = T_0(P)$, and if the Jacobian of T_0 at P is different from zero, then there exist a radius r and an n_0 such that the sphere of radius r with center at P_0 is contained in $T_n(D)$, $n \geq n_0$.

Writing

$$P: \qquad (x_1^0, \cdots, x_k^0); \qquad T_n(P): \qquad (u_1^n, \cdots, u_k^n)$$

we put for each and every n,

$$x_j = x_j^0 + \xi_j, \qquad u_j = u_j^n + \omega_j$$

and we consider the system of equations

$$F_j(\xi_1, \cdots, \xi_k; \omega_1, \cdots, \omega_k) \equiv f_j^n(x_p^0 + \xi_p) - (u_j^n + \omega_j) = 0$$

With ξ, ω in the place of x, u the assumptions of Lemma 1 are fulfilled, including assumption (16) for $n \geq n_1$. Thus a sphere

$$\Sigma_{j=1}^k \omega_j^2 = \Sigma_{j=1}^k (u_j - u_j^n)^2 < r_0^2$$

of fixed radius r_0 is contained in $T_n(D)$, $n \geq n_1$. Since $T_n(P) \to P_0$ as $n \to \infty$, the conclusion of our theorem will hold for $r = r_0/2$, say, and some $n_0 \geq n_1$.

Theorem 4. *If* (12) *is a transformation with property* (14) *of any domain D containing the origin into part of itself, and if the sequence of iterates*

$$(17) \qquad \{T, T^2, T^3, \cdots\}$$

is majorized, then the transformation T is an automorphism.

We will first prove that T is a one-one transformation of D into part of itself. Consider any two points P, Q which T maps into the same point $T(P) = T(Q)$. This implies

$$(18) \qquad T^n(P) = T^n(Q), \qquad n = 1, 2, \cdots$$

However, by Theorem 5, Chapter I, there exists a sequence $\{T^{n_\nu}\}$ which converges (weakly) to the identity transformation. But, since the sequence $\{T^{n_\nu}\}$ is majorized in D, and hence bounded in

every compact set S of D, it follows from the italicized statement preceding Theorem 2 that the sequence $\{T^{n_s}\}$ converges continuously in D to the identity transformation. From the mere convergence of $\{T^{n_s}\}$ to the identity we conclude that

$$T^{n_s}(P) \to P, \qquad T^{n_s}(Q) \to Q$$

Thus (18) implies $P = Q$, and hence T is a one-one map of D.

We are now going to apply Theorem 3. The point set $T^{n_s}(D)$ is contained in $T(D)$, T^{n_s} converges continuously in D to a transformation with nowhere vanishing Jacobian, and $T^{n_s}(P) \to P$ for each P in D. Hence for each P in D there exists an open sphere which contains P and which in turn is contained in all $T^{n_s}(D)$ for $s \geq s_0$. Thus there exists an open sphere which contains P and which in turn is contained in $T(D)$. Therefore $T(D)$ cannot be a proper part of D, and the proof of the theorem is completed.

Corollary to Theorem 4. *If D is a domain in the space of complex variables containing the origin, if D is bounded, and if* (12) *is a transformation of D into part of itself with property* (14), *then T is an automorphism of D.*

If D is not bounded but possesses the alternative property stated in Theorem 2, *and if* (12) *is a transformation of D into part of itself with property* (14) *for which the linear parts $\{L_T, L_{T^2}, L_{T^3}, \cdots\}$ are bounded, then T is an automorphism.*

In fact, if D is bounded, the sequence (17) is automatically bounded; in the more general case, the sequence (17) is bounded in virtue of Theorem 10, Chapter I.

§4. Groups of Automorphisms with Fixed Point

Our next task will be to substantiate Theorem 8, Chapter I concerning groups of transformations. We first observe that if a group of transformations $\{T(\alpha)\}$ is not only weakly bounded but also majorizable then the transformation

$$S = \int_G L(T(\alpha^{-1})) \cdot T(\alpha) d\mu(\alpha) \tag{19}$$

which carries the group into a similar group of *linear* transformations, is represented by power-series which in the neighborhood of the origin have a domain of convergence. In fact, if the family $\{T(\alpha)\}$ is majorized then the family $\{L(T(\alpha^{-1}))\}$ is majorized and hence the family $\{L(T(\alpha^{-1})) \cdot T(\alpha)\}$ is majorized. And now we have only to observe that if

$$|c_{n_1 \ldots n_k}(\alpha)| \leq A_{n_1 \ldots n_k}$$

then by property (i) of our group measure given in section 6, Chapter I, we also have

(20) $$\left|\int_G c_{n_1\cdots n_k}(\alpha)d\mu(\alpha)\right| \leq A_{n_1\cdots n_k}$$

Thus Theorem 8, Chapter I leads to the following conclusion. If $\{T(\alpha)\}$ is a group of automorphisms of a domain D with the origin as fixed point, and if the group is majorized (for instance, for complex variables, if D is bounded) then, geometrically speaking, there exists a change of coordinates in the neighborhood of the origin, such that the given automorphisms are linear transformations in the new coordinates. This suggests generalizing our theorem to domains D in general coordinate spaces in which changes of coordinates are intrinsic to the definition of space.

A (real) analytic space Σ_k is defined as follows.

I. Σ_k is a connected point set of dimension k in which a topology is defined by open sets.

II. There exists a covering of Σ_k by open sets U such that each U is the homeomorphic image of a domain V in a fixed Euclidean E_k: $(x_1, \cdots, x_k)$. These mappings $f: V \to U$ assign to each point of U a set of parameters $(x_1, \cdots, x_k)$.

III. If the intersection $U_1 \cap U_2$ is not empty, then the points of this intersection receive parameters $(x_1, \cdots, x_k)$ by $f_1: V_1 \to U_1$, and parameters $(x'_1, \cdots, x'_k)$ by $f_2: V_2 \to U_2$. Thus there is defined a mapping $f_2^{-1}f_1$ in E_k of a subset of V_1 onto a subset of V_2 by equations $x'_j = x'_j(x_1, \cdots, x_k)$. It is required that all these mappings in each connected component of the subset involved shall be analytic, and that the Jacobian $\partial(x'_1, \cdots, x'_k)/\partial(x_1, \cdots, x_k)$ be positive at every point where the mapping is defined.

In addition to the mappings just described, consider any other homeomorphism $f': V' \to U'$ between a domain V' in E_k and a domain U' in Σ_k which is related to the mappings f in the manner described. All these mappings (including the mappings f) are called *allowable mappings*, and the resulting parameters on some domains in Σ_k are called *allowable coordinates.*

For a *complex* analytic space Σ_{2k}, we define I and II as before and modify III in the following way.

III (complex). Denoting the coordinates of V_1 by $x_1, y_1, x_2, y_2, \cdots, x_k, y_k$ and of V_2 by $x'_1, y'_2, \cdots, y'_k$, and putting $z_j = x_j + iy_j$, $z'_j = x'_j + iy'_j$ then the homeomorphism $f_2^{-1}f_1$ shall be expressible by analytic functions $z'_j = z'_j(z_1, \cdots, z_k)$.

A function in a domain of an analytic space is *analytic* if in the

neighborhood of any point of the domain it is an analytic function of allowable coordinates. More generally, a transformation T from a domain of one analytic space into another analytic space (of equal or different dimensions) is *analytic* if, P_0 being a point of the domain, and Q_0 its image under T, there exist a neighborhood $N(P_0)$ with allowable coordinates $x_1, \cdots, x_k$, and a neighborhood $N(Q_0)$ with allowable coordinates $x'_1, \cdots, x'_l$ such that (i) the image of $N(P_0)$ under T lies in $N(Q_0)$ and (ii) the transformation of $N(P_0)$ into $N(Q_0)$ can be expressed by analytic functions $x'_\lambda = f_\lambda(x_1, \cdots, x_k)$.

Theorem 8, Chapter I gives the following theorem.

Theorem 5. *If Σ_{2k} is a complex analytic coordinate space, and D is a domain in Σ_{2k}, then corresponding to any compact group of automorphisms of D with a point O as fixed point there exist allowable coordinates with O as origin, such that in this coordinate system the given automorphisms are all linear transformations.*

§5. Analytic Mapping of Spherical Surfaces onto Each Other

We consider a finite number of power series

$$(21) \qquad f_j(z) = \Sigma_{n_1, \cdots, n_k = 0}^{\infty} a_{n_1 \cdots n_k}^{(j)} z_1^{n_1} \cdots z_k^{n_k}, \qquad j = 1, \cdots, h$$

$(z_\nu = x_\nu + iy_\nu)$ assumed convergent in a fixed sphere

$$S_R\colon |z_1|^2 + \cdots + |z_k|^2 < R^2.$$

Our purpose in this section is to derive a set of necessary and sufficient conditions on the functions $f_j(z)$ in order that the mapping

$$w_j = f_j(z_1, \cdots, z_k), \qquad j = 1, \cdots, h$$

shall carry each spherical hypersurface $|z_1|^2 + \cdots + |z_k|^2 =$ constant $(< R^2)$ into points of some spherical hypersurface $|w_1|^2 + \cdots + |w_h|^2 =$ constant. For this purpose we form the sum

$$(22) \qquad F(x, y) = \Sigma_{j=1}^{h} |f_j(z)|^2$$

Since

$$(23) \qquad \overline{f_j(z)} = \Sigma_n \overline{a_{n_1 \cdots n_k}^{(j)}} \bar{z}_1^{n_1} \cdots \bar{z}_k^{n_k}$$

we obtain, on introducing the numbers

$$(24) \qquad C_{m_1 \cdots m_k, n_1 \cdots n_k} = \Sigma_{j=1}^{h} a_{m_1 \cdots m_k}^{(j)} \overline{a_{n_1 \cdots n_k}^{(j)}}$$

the expansion

$$(25) \qquad F(x, y) = \Sigma_{m,n} C_{m,n} z_1^{m_1} \bar{z}_1^{n_1} \cdots z_k^{m_k} \bar{z}_k^{n_k}$$

the latter "double" series being uniformly absolutely convergent. Putting

(26) $$z_1 = r_1 e^{i\varphi_1}, \quad \cdots, \quad z_k = r_k e^{i\varphi_k}; \qquad r_\nu = |z_\nu|$$

we obtain

$$F(x, y) = \Sigma_{m,n} C_{m,n} r_1^{m_1+n_1} \cdots r_k^{m_k+n_k} e^{i((m_1-n_1)\varphi_1 + \cdots + (m_k-n_k)\varphi_k)}$$

Introducing the numbers $\delta_1 = m_1 - n_1, \cdots, \delta_k = m_k - n_k$ and collecting terms suitably we hence obtain a series

$$\Sigma_{\delta_1, \cdots, \delta_k = -\infty}^{\infty} B_\delta(r_1, \cdots, r_k) e^{i(\delta_1\varphi_1 + \cdots + \delta_k\varphi_k)}$$

This series being uniformly convergent in all variables, it is the multiple Fourier series of its sum function which is a periodic function with period 2π in each variable $\varphi_1, \cdots, \varphi_k$.

We now assume that $F(x, y)$ depends only on $r_1, \cdots, r_k$, thus being constant in $\varphi_1, \cdots, \varphi_k$. In this case only the constant term of the Fourier series is $\neq 0$, and thus we obtain $B_{\delta_1 \cdots \delta_k} = 0$ for $\delta_1^2 + \cdots + \delta_k^2 > 0$. Since $m_\nu = n_\nu + \delta_\nu$, we have

$$B_\delta = \Sigma_n C_{n+\delta,n} r_1^{n_1+2\delta_1} \cdots r_k^{n_k+2\delta_k}$$

the sum extending over those nonnegative values of $n_1, \cdots, n_k$ for which the sums $n_\nu + \delta_\nu$ are ≥ 0. The series is absolutely convergent for $r_1^2 + \cdots + r_k^2 < R^2$ and it can vanish only if all its coefficients vanish. Thus we obtain

(27) $$C_{m_1 \cdots m_k, n_1 \cdots n_k} = 0 \qquad \text{if} \qquad (m_1 - n_1)^2 + \cdots + (m_k - n_k)^2 > 0$$

and hence

(28) $$F(x, y) \equiv \Sigma_n C_{n,n} r_1^{2n_1} \cdots r_k^{2n_k}$$

We now further restrict $F(x, y)$ to have a value which depends only on $r_1^2 + \cdots + r_k^2 = r^2$. Introducing the quantities $t_1 = r_1^2, \cdots, t_k = r_k^2$, $t = r^2$, we have

$$F(x, y) = \Sigma_n C_{n,n} t_1^{n_1} \cdots t_k^{n_k}$$

and this represents an analytic function of the variables $t_1, \cdots, t_k$ in the neighborhood of the origin. Replacing the variable t_1 by $t = t_1 + \cdots + t_k$ and leaving $t_2, \cdots, t_k$ unchanged we obtain a function $G(t; t_2, \cdots, t_k)$ which again is analytic for small values of its variables. By our assumption on $F(x, y)$, if t has any fixed (small) positive value, then $G(t; t_2, \cdots, t_k)$ is constant for $t_2 \geq 0, \cdots,$

$t_k \geq 0,\ t_2 + \cdots + t_k < t$; hence, by analytic continuation, it is identically constant in the variables $t_2, \cdots, t_k$ for each positive value of t. In particular $G(t; t_2, \cdots, t_k) \equiv G(t; 0, \cdots, 0)$, for small positive t. Now for each $(t_2, \cdots, t_k)$ both sides are analytic in t, and therefore this identity is valid for all small t. Now the function $G(t; 0, \cdots, 0)$ has some power series $\Sigma_{N=0}^{\infty} A_N t^N$, and therefore we obtain

$$\Sigma_n C_{n,n} t_1^{n_1} \cdots t_k^{n_k} \equiv \Sigma_{N=0}^{\infty} A_N (t_1 + \cdots + t_k)^N$$

In other words there exists a sequence of constants A_N such that

$$C_{n_1 \cdots n_k, n_1 \cdots n_k} = \frac{N!}{n_1! \cdots n_k!} A_N, \qquad N = n_1 + \cdots + n_k \tag{29}$$

Thus we have reached the following result.

Theorem 6. If

$$w_j = f_j(z_1, \cdots, z_k), \qquad j = 1, \cdots, h \tag{30}$$

are h functions with power-series (21) *convergent inside some sphere S_R, then the sum*

$$|w_1|^2 + \cdots + |w_h|^2 \equiv |f_1(z)|^2 + \cdots + |f_h(z)|^2 \tag{31}$$

is dependent on

$$r^2 = |z_1|^2 + \cdots + |z_k|^2 \tag{32}$$

that is, the transformation (30) *carries each spherical hyper-surface* $r =$ const. *into points of some spherical hypersurface*

$$|w_1|^2 + \cdots + |w_h|^2 = \text{const.} \tag{33}$$

if and only if the following two sets of conditions are satisfied:

$$\Sigma_{j=1}^{h} a_{m_1 \cdots m_k}^{(j)} \overline{a_{n_1 \cdots n_k}^{(j)}} = 0 \qquad \text{for} \quad (m_1 - n_1)^2 + \cdots + (m_k - n_k)^2 > 0 \tag{34}$$

and

$$\Sigma_{j=1}^{h} |a_{n_1 \cdots n_k}^{(j)}|^2 = \frac{(n_1 + \cdots + n_k)!}{n_1! \cdots n_k!} A_{n_1 + \cdots + n_k} \tag{35}$$

where A_N, $N = 0, 1, \cdots$, are suitable nonnegative numbers.

Introducing for each multi-index $n_1, \cdots, n_k$ the vector $\mathfrak{a}_{n_1 \cdots n_k}$ with the h components $a_{n_1 \cdots n_k}^{(j)}$, relation (34) requires that any two of these vectors shall be (unitary) orthogonal, and relation (35)

implies in particular that all vectors for which $n_1 + \cdots + n_k$ has the same value N shall be either all 0 or all $\neq 0$. On the other hand, in the space of h components there can be at most h nonvanishing vectors which are mutually orthogonal. Thus the number s of nonvanishing vectors must be a number $\leq h$ and s must be a sum of some of the following numbers

$$(36) \qquad 1, \quad \frac{k}{1}, \quad \frac{k(k+1)}{1 \cdot 2}, \quad \cdots, \quad \frac{k(k+1)\cdots(k+N-1)}{1 \cdot 2 \cdots N}, \quad \cdots$$

no repetitions being allowed.

Thus if $k \geq 2$ and $h < k$ we can only have $s = 1$ which leads to the trivial transformation $w_j = c_j$. Thus, but for that trivial transformation, there is no transformation as described in our theorem from a space into a lower-dimensional one. The case $h = k$ is very interesting since it establishes a sharp difference between the classical case $k = 1$ and all other cases $k \geq 2$. For $k = 1$ all numbers (36) have the value 1 and all transformations $w = cz^N$ $(N = 0, 1, \cdots)$ are admissible. However, for $k \geq 2$, the numbers (36) from the third onward are greater than the common value of $h = k$ and thus the only possibility is $s = 1$ or $s = k$. The first is the previous trivial transformation $w_j = c_j$, whereas the second is the important class of transformations

$$(37) \qquad w_j = \Sigma_{m=1}^{k} a_m^{(j)} z_m, \qquad j = 1, \cdots, k$$

with

$$(38) \qquad \Sigma_{j=1}^{k} a_m^{(j)} \overline{a_n^{(j)}} = \begin{cases} 0 & \text{if} \quad m \neq n \\ A(\neq 0) & \text{if} \quad m = n \end{cases}$$

Dividing $a_m^{(j)}$ by $\sqrt{A}$ we obtain constants b_m^i with the property

$$(39) \qquad \Sigma_{j=1}^{k} b_m^j \overline{b_n^j} = \begin{cases} 0 & \text{if} \quad m \neq n \\ 1 & \text{if} \quad m = n \end{cases}$$

A matrix $[b_m^j]$ of this description is a so-called unitary matrix. Linear transformations with unitary matrices are topological transformations which preserve the Euclidean distance from the origin. Thus we have the following result: *For $k \geq 2$ the only transformations as described in Theorem 6 which carry a neighborhood of the origin into points of the same space and which do not map all points into one*

point are (topological) linear transformations of the form (37) *with* (38) *holding for some constant* $A \neq 0$. *If in addition we ask that the Euclidean distance from the origin is preserved, then the transformation is a unitary linear transformation.*

The case $h > k$ admits nonlinear polynomial transformations; however, for $k \geq 2$, the degree of the polynomials is bounded in terms of h (and k), but not for $k = 1$.

§6. Schwarz's Lemma and the Hadamard Three-spheres Theorem

If $G(t)$ is analytic in the closed unit circle $|t| \leq 1$ and if $G(0) = 0$, then Schwarz's lemma states that

$$|G(t)| \leq |t| \max_{0 \leq \vartheta \leq 2\pi} |G(e^{i\vartheta})|, \qquad |t| \leq 1 \tag{40}$$

the equality sign holding if and only if $G(t) = ct$, where $|c| = 1$. For any function $G(t)$ analytic in the closed unit circle, not necessarily vanishing at $t = 0$, Hadamard's three-circles theorem states that

$$\{\max_{|t|=\rho_2} |G(t)|\}^{\alpha_2} \leq \{\max_{|t|=\rho_1} |G(t)|\}^{\alpha_1} \cdot \{\max_{|t|=\rho_3} |G(t)|\}^{\alpha_3} \tag{41}$$

for any ρ's in

$$0 < \rho_1 < \rho_2 < \rho_3 \leq 1 \tag{42}$$

where

$$\alpha_1 = \log \frac{\rho_3}{\rho_2}, \qquad \alpha_2 = \log \frac{\rho_3}{\rho_1}, \qquad \alpha_3 = \log \frac{\rho_2}{\rho_1} \tag{43}$$

These results have (easily derivable) parallels in several variables. Let $F(z_1, \cdots, z_k)$ be analytic in the closed unit hypersphere

$$|z_1|^2 + \cdots + |z_k|^2 \leq 1 \tag{44}$$

and denote by

$$M(\rho) = \max_{||z||=\rho} |F(z_1, \cdots, z_k)| \tag{45}$$

where

$$||z|| = [|z_1|^2 + \cdots + |z_k|^2]^{\frac{1}{2}} \tag{46}$$

If we introduce the function

$$G(t; z) = F(z_1 t, \cdots, z_k t) \tag{47}$$

then

$$M(\rho) = \max_{||z||=1} M(\rho; z) \qquad (0 < \rho \leq 1) \tag{48}$$

where

(49) $$M(\rho; z) = \max_{|t|=\rho} |G(t; z)|$$

If $F(z)$ vanishes at $z_1 = \cdots = z_k = 0$, then $G(0, z) = 0$ for every z in (44), and for each fixed z^0 on the boundary of (44), ($||z^0|| = 1$), Schwarz's lemma yields

$$M(\rho; z^0) \leq \rho M(1; z^0) \qquad (\rho \leq 1)$$

By (48) this yields (since $||z^0|| = 1$)

$$M(\rho;\ z^0) \leq \rho M(1)$$

and since this holds for every z^0 for which $||z^0|| = 1$, we have finally

(50) $$M(\rho) \leq \rho M(1) \qquad (\rho \leq 1)$$

which yields a form of Schwarz's lemma for $F(z_1, \cdots, z_k)$.

If F is merely analytic in (44) (not necessarily vanishing at $(z) = (0)$) we obtain by (41)

$$[M(\rho_2; z^0)]^{\alpha_2} \leq [M(\rho_1; z^0)]^{\alpha_1}[M(\rho_3; z^0)]^{\alpha_3}$$

for every z^0 on $||z^0|| = 1$, and the ρ's and α's as in (42), (43). Using (48) we conclude that

(51) $$M(\rho_2)^{\alpha_2} \leq M(\rho_1)^{\alpha_1} M(\rho_3)^{\alpha_3}$$

a form of Hadamard's three-spheres theorem for F.

The inequalities (50) and (51) take on especial interest when we consider several functions. Let

(52) $$f_1(z_1, \cdots, z_k), \qquad \cdots, \qquad f_n(z_1, \cdots, z_k)$$

be n functions analytic in the unit hypersphere (44), $n \gtreqless k$, and let $a_1, \cdots, a_n$ be n arbitrary complex constants. Later we shall choose these constants in an advantageous manner. Define

(53) $$F_a(z) = a_1 f_1(z) + \cdots + a_n f_n(z)$$

Then $F_a(z)$ is analytic in (44). If we assume further that each $f_1, \cdots, f_n$ vanishes at the origin, then Schwarz's lemma (50) yields

(54) $$\max_{||z||=\rho} |a_1 f_1(z) + \cdots + a_n f_n(z)| \\ \leq \rho \ \max_{||z||=1} |a_1 f_1(z) + \cdots + a_n f_n(z)|$$

For p any real number greater than unity, Hölder's inequality states

(55) $$|a_1 f_1(z) + \cdots + a_n f_n(z)| \leq ||a||_q \cdot ||f(z)||_p$$

where

$$(56)\quad ||a||_q = [|a_1|^q + \cdots + |a_n|^q]^{\frac{1}{q}};$$
$$||f||_p = [|f_1|^p + \cdots + |f_n|^p]^{\frac{1}{p}}; \qquad \frac{1}{p} + \frac{1}{q} = 1$$

Inserting (55) into the right-hand side of (54), we have

$$(57)\quad \max_{||z||=\rho} |a_1 f_1(z) + \cdots + a_n f_n(z) \leq \rho ||a||_q \cdot \max_{||z||=1} ||f(z)||_p$$

Now denote by (z') a value of (z) on $||z|| = \rho$ for which $||f(z)||_p$ attains its maximum on $||z|| = \rho$, so that

$$(58)\qquad ||f(z)||_p \leq ||f(z')||_p \qquad \text{for} \qquad ||z|| = \rho$$

In terms of this point (z') choose

$$(59)\qquad a_\mu = \begin{cases} \dfrac{|f_\mu(z')|^p}{f_\mu(z')} & \text{if} \quad f_\mu(z') \neq 0 \\ 0 \quad \text{if} & f_\mu(z') = 0 \end{cases}$$

With these values of $a_1, \cdots, a_n$ we have

$$||a||_q = [\Sigma_{\mu=1}^n |f_\mu(z')|^{(p-1)q}]^{\frac{1}{q}} = [\Sigma_{\mu=1}^n |f_\mu(z')|^p]^{\frac{1}{q}}$$

and

$$a_1 f_1(z') + \cdots + a_n f_n(z') = \Sigma_{\mu=1}^n |f_\mu(z')|^p$$

On inserting these values into (57), we have

$$\frac{\Sigma_{\mu=1}^n |f_\mu(z')|^p}{[\Sigma_{\mu=1}^n |f_\mu(z')|^p]^{\frac{1}{q}}} \leq \rho \cdot \max_{||z||=1} ||f(z)||_p$$

Since $1 - 1/q = 1/p$, this yields

$$(60)\qquad ||f(z')||_p \leq \rho \cdot \max_{||z||=1} ||f(z)||_p$$

Finally by (58) we conclude

$$(61)\quad \max_{||z||=\rho} ||f(z)||_p \leq \rho \cdot \max_{||z||=1} ||f(z)||_p, \qquad (0 \leq \rho \leq 1)$$

We state this form of Schwarz's theorem as a theorem.

Theorem 7. *Let $f_1(z_1, \cdots, z_k), \cdots, f_n(z_1, \cdots, z_k)$ be n, $(n \gtreqless k)$, functions each analytic in the closed unit hypersphere (44), and each vanishing at the origin. Then for any value of $p > 1$ we have*

$$(62)\quad [|f_1(z)|^p + \cdots + |f_n(z)|^p]^{\frac{1}{p}} \leq [|z_1|^2 + \cdots + |z_k|^2]^{\frac{1}{2}} \cdot \max_{|w_1|^2+\cdots+|w_k|^2=1} [|f_1(w)|^p + \cdots + |f_n(w)|^p]^{\frac{1}{p}}$$

for every value of z *in* (44).

We now drop the hypothesis that the f's vanish at the origin, retaining only the hypothesis that they are analytic in (44). Then by Hadamard's three-spheres theorem (51) we have

$$\max_{\|z\|=\rho_2} |a_1f_1(z) + \cdots + a_nf_n(z)| \leq [\{\max_{\|z\|=\rho_1} |a_1f_1(z) + \cdots + a_nf_n(z)|\}^{\frac{\alpha_1}{\alpha_2}} \cdot \{\max_{\|z\|=\rho_3} |a_1f_1(z) + \cdots + a_nf_n(z)\}^{\frac{\alpha_3}{\alpha_2}}]$$

On using Hölder's inequality (55) this yields

$$(63)\quad \max_{\|z\|=\rho_2} |a_1f_1(z) + \cdots + a_nf_n(z)| \leq \|a\|_q^{\frac{\alpha_1+\alpha_3}{\alpha_2}} \cdot \max_{\|z\|=\rho_1} \|f(z)\|_p^{\frac{\alpha_1}{\alpha_2}} \cdot \max_{\|z\|=\rho_3} \|f(z)\|^{\frac{\alpha_3}{\alpha_2}}$$

On noting that $(\alpha_1 + \alpha_3)/\alpha_2 = 1$ by (43), we have from (63)

$$\frac{\max_{\|z\|=\rho_2} |a_1f_1(z) + \cdots + a_nf_n(z)|}{\|a\|_q} \leq \max_{\|z\|=\rho_1} \|f(z)\|_p^{\frac{\alpha_1}{\alpha_2}} \cdot \max_{\|z\|=\rho_3} \|f(z)\|_p^{\frac{\alpha_3}{\alpha_2}}$$

Repeating the argument used in passing from (55) to (61), with ρ replaced by ρ_2, we conclude that

$$(64)\quad \max_{\|z\|=\rho_2} \|f(z)\|_p \leq \max_{\|z\|=\rho_1} \|f(z)\|_p^{\frac{\alpha_1}{\alpha_2}} \cdot \max_{\|z\|=\rho_3} \|f(z)\|_p^{\frac{\alpha_3}{\alpha_2}}$$

which yields the following form of the three-spheres theorem.

Theorem 8. *Let* $f_1(z_1, \cdots, z_k), \cdots, f_n(z_1, \cdots, z_k)$ *be* n, $(n \gtreqless k)$, *functions each analytic in the closed unit hypersphere* (44). *Then for any value of* $p > 1$ *we have* (64) *holding, where the* ρ*'s and* α*'s are as in* (42) *and* (43), *and where the norms* $\|z\|$, $\|f\|_p$ *are defined in* (46), (56).

This of course yields the result that the function

$$\log \max \|f(z)\| \qquad \text{for} \qquad \|z\| = \rho$$

is a convex function of $\log \rho$.

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3. P. Fatou, Sur certaines fonctions uniformes de deux variables, *C. R. Acad. Sci. Paris*, Vol. 175 (1922), pp. 1030–1033.
4. D. C. May, Jr., An integral formula for analytic functions of k variables with some applications, Princeton University (unpublished) dissertation, 1941.

Furthermore, in connection with section 5, Theorem 6, see:

5. S. Bochner, Curvature in Hermitian metric, *Bull. Amer. Math. Soc.*, Vol. 53 (1947), pp. 179–195.

where the method of "unitary vectors" is used to prove that the hyperbolic line-element in one complex variable cannot be isometrically (complex-analytically) imbedded into a domain of any finite number of complex variables.

CHAPTER IV

Analytic Completion

Any domain in one complex variable may be the natural domain of analyticity for some suitable function. However, in the space of variables $(z; u_1, \cdots, u_n) z = x + iy$, u_m real, $n \geq 1$, there are pairs of domains D, $\tilde{D}$, with $\tilde{D} \supset D$, such that any function which is analytic in D has an analytic continuation in $\tilde{D}$. This phenomenon of analytic continuation which refers to the domain of definition and not to a single function will be termed analytic *completion*, and the domain $\tilde{D}$ will be called an *analytic completion* of D. Some of the more general theorems on completion will be the object of the present chapter. These theorems will depend upon the geometric structure of the domain D.

We can derive immediately a very simple but general property of analytic completions, namely:

General property of analytic completions. If D is a domain in $(z; u_1, \cdots, u_n)$-space with an analytic completion D', and if $f(z; u_1, \cdots, u_n)$ is analytic in D (and hence also in D') then f assumes in D every value which it assumes in D'. In particular if f is bounded in D then its analytic continuation is bounded in D', and with the same bound. To see this assume for the moment that f assumes in D' a value v which it does not assume in D. Then the function $1/(f - v)$ is analytic in D and hence by analytic continuation also in D', which is a contradiction.

The method of the present chapter is based upon Cauchy's integral formula in one complex variable. For this method we will always have at least one complex variable and at least one other variable which may be either real or complex. Not only does the method require that at least one of the variables be complex, but also the results themselves can be shown not to be true for the case in which all of the variables are real (cf. section 9).

A very easy example will serve to illustrate the general method used.

Example. *Let $f(z, u)$ be analytic in the spherical shell*

$$S: \qquad (R - \epsilon)^2 < |z|^2 + u^2 < (R + \epsilon)^2, \qquad (0 < \epsilon < R)$$

Then f has an analytic continuation into the sphere

$$\tilde{S}: \qquad |z|^2 + u^2 < (R + \epsilon)^2$$

Thus $\tilde{S}$ is an analytic completion of S.

Proof. For each fixed value of u in

$$-R < u < R$$

define

$$\varphi(z, u) = \frac{1}{2\pi i} \int_{C(u)} \frac{f(t, u)}{t - z}\, dt$$

where $C(u)$ is the circle

$$|t| = [R^2 - u^2]^{\frac{1}{2}}$$

Since $f(t, u)$ is a continuous (even analytic) function of t for t on the contour $C(u)$ it follows that $\varphi(z, u)$ is an analytic function of z for z in the interior of the circle $C(u)$,

$$|z| < [R^2 - u^2]^{\frac{1}{2}}$$

We next observe that since $f(t, u)$ is analytic in the shell S we may shift the contour $C(u)$ to any simple closed contour contained within the annulus

$$[(R - \epsilon)^2 - u^2]^{\frac{1}{2}} < |t| < [(R + \epsilon)^2 - u^2]^{\frac{1}{2}}$$

(u fixed in $-R < u < R$), and not change the value of the resulting function $\varphi(z, u)$. We will use this fact a little later.

Now let u_0 be a fixed value of u in

$$-R < u_0 < R$$

and let us form the integral

$$I = \frac{1}{2\pi i} \int_{C(u_0)} \frac{f(t, u)}{t - z}\, dt$$

where the contour $C(u_0)$ is fixed but where we now let u vary over an interval

$$u_0 - \epsilon < u < u_0 + \epsilon$$

Since $f(z, u)$ is given to be analytic in (z, u) in the shell S, and since the point set

$$[t \epsilon C(u_0),\ u_0 - \epsilon < u < u_0 + \epsilon]$$

lies in S, it follows easily that the integral (I) defines a function which is analytic in (z, u) in the domain

$$[|z|^2 < R^2 - u_0^2,\ u_0 - \epsilon < u < u_0 + \epsilon]$$

We now make two simple observations. First, the function defined by the integral I and the function $\varphi(z, u)$ are identical for $u = u_0$, as one sees by comparing their definitions. Secondly, by the remark made in the preceding paragraph, the function defined by I and the function $\varphi(z, u)$ actually agree for all values of u in $u_0 - \epsilon < u < u_0 + \epsilon$ since for such values of u the contour $C(u_0)$ serves as an admissible contour. Hence $\varphi(z, u)$ is analytic in (z, u) throughout the entire sphere $|z|^2 + u^2 < R^2$.

It remains to identify φ with f. This is very easily done. Consider for example a value of u in, say,

$$R - \epsilon < u < R - \frac{\epsilon}{2}$$

For such a value of u the function $f(t, u)$ is analytic, not just in a neighborhood of the contour $C(u)$ but even throughout its interior. Hence for such values of u we have $\varphi(z, u) \equiv f(z, u)$, and hence $\varphi(z, u)$ furnishes the desired analytic continuation of $f(z, u)$ into $|z|^2 + u^2 < R^2$. This yields the desired result.

Two things are relevant in this example, one is that for each level u there is a contour over which we can integrate, and the second is that in the neighborhood of some point we can identify φ with f.

§1. PRELIMINARIES

For $z = x + iy$ we consider the Euclidean plane of finite z-values; the point $z = \infty$ is not included. Any Jordan curve C has an interior and an exterior. The interior will be denoted by R_C; it is a bounded domain whose boundary is C. The closure of R_C will be denoted by $\bar{R}_C$.

An open set need not be a domain but it decomposes uniquely into a finite or denumerable number of mutually disjoint domains. They will be called its components. If Δ is any open set, then the union of the domains R_C for all C in Δ will be called the (geometric) *completion* of Δ and denoted by $\tilde{\Delta}$. Since each C must be contained in one component, the completion of Δ is the union of the completions of its components. However, if Δ^1 and Δ^2 are disjoint domains, then $\tilde{\Delta}^1$ and $\tilde{\Delta}^2$ are either disjoint or one is contained in the other. For instance, if Δ^1 is $|z| < 1$ and Δ^2 is $2 < |z| < 3$, then $\tilde{\Delta}^1$ is $|z| < 1$ and $\tilde{\Delta}^2$ is $|z| < 3$.

Any Jordan curve C lying in a domain will be called a *retrosection* of the domain. A retrosection C of a domain Δ will be called a *complete* retrosection if the union of Δ and R_C is $\tilde{\Delta}$. A domain need not have a

complete retrosection, for instance if it is the unit circle after deletion of a sequence of points which converge to a boundary point.

Now let D be any domain in the $(n+2)$-dimensional space $(z;\ u_1, \cdots, u_n)$. Its projection on $z = 0$ is a domain $U = U_D$ in $(u_1, \cdots, u_n)$, and for each $c = (c_1, \cdots c_n)$ in U, the intersection of D with the 2-dimensional space

$$(1)\qquad u_1 = c_1, \qquad \cdots, \qquad u_n = c_n$$

is a 2-dimensional open set whose projection on the z-plane will be denoted by $\Delta(u)$. Thus D is the point set

$$(2)\quad D: \qquad [u \varepsilon U; \qquad z \varepsilon \Delta(u)]$$

We now form the larger point set

$$(3)\quad \tilde{D}_z: \qquad [u \varepsilon U; \qquad z \varepsilon \tilde{\Delta}(u)]$$

where $\tilde{\Delta}(u)$ is the 2-dimensional completion of $\Delta(u)$. We will call (3) the *z-completion* of (2). Our first task is to show that (3) is again a domain.

Let $P(c; z_0)$ be a point of $\tilde{D}_z$. By definition of $\tilde{\Delta}(u)$ there exists a retrosection C in $\Delta(c)$ such that $z_0 \varepsilon R_C$. Now the curve

$$(4)\qquad [u = c; \zeta \varepsilon C]$$

is a bounded closed point set in D. Hence there exists a neighborhood

$$(5)\qquad U_{c,\rho}:\ (u_1 - c_1)^2 + \cdots + (u_n - c_n)^2 < \rho^2 \qquad (\rho > 0)$$

in U, such that the point set

$$(6)\qquad [u \varepsilon U_{c,\rho}; \zeta \varepsilon C]$$

is also contained in D. Therefore, C is a retrosection in $\Delta(u)$ for $u \varepsilon U_{c,\rho}$, and hence the point set

$$(7)\qquad [u \varepsilon U_{c,\rho}; z \varepsilon \bar{R}_C]$$

belongs to $\tilde{D}_z$. In particular, the point (c, z_0) from which we started is an interior point of $\tilde{D}_z$. Also, there exists a point ζ_0 on C such that the points (c, ζ_0) and (c, z_0) can be connected in $\tilde{D}_z$, and (c, ζ_0) belongs to D proper. Similarly any other point (c', z_0') in $\tilde{D}_z$ can be connected with a point (c', ζ_0') in D, and since any two points of D are connected, we obtain the result that *the z-completion of a domain is again a domain.*

Now assume that Δ has a finite number of components, $\Delta = (\Delta^1, \cdots, \Delta^r)$, with disjoint completions and complete retrosections. If C^p, $p = 1, \cdots, r$, is a complete retrosection of Δ^p, then we call the

system of curves $C = (C^1, \cdots, C^r)$ a complete retrosection of Δ, and the pointset R_C shall be the union $\Sigma_{p=1}^{r} R_{C^p}$. An arbitrary retrosection C of Δ shall be any system of curves $(C^{p_1}, C^{p_2}, \cdots)$ in Δ, and R_C shall be the corresponding subset of $\tilde{\Delta}$.

§2. THEOREMS ON COMPLETION

Theorem 1. *If an arbitrary domain (2) is such that for each $u \varepsilon U$ the number of components $\{\Delta^p(u)\}$ is finite; if each component $\Delta^n(u)$ of $\Delta(u)$ has a complete retrosection, and any two completions $\tilde{\Delta}^p(u)$, $\tilde{\Delta}^q(u)$ are disjoint for $p \neq q$; if corresponding to each $P(c,z_0)$ of $\tilde{D}(z)$ there exists a retrosection C_0 of $\Delta(c)$ such that for every u of a neighborhood (5), $P(u,z_0)$ is contained in R_{C_0} and C_0 is topologically homologous to some complete retrosection $C(u)$ in $\Delta(u)$; and if for at least one point u, every component of $\Delta(u)$ is simply connected, then (3) is an analytic completion of (2). Every function $f(z; u_1, \cdots, u_n)$ which is defined and analytic in D has a unique analytic continuation in $\tilde{D}_z$.*

In preparation for the proof we will first make a few observations about domains and functions in the z-plane.

If a set Δ in the z-plane has a complete retrosection C then there exists a sequence of complete retrosections $C_1, C_2, \cdots$ such that $\bar{R}_{C_n} \subset R_{C_{n+1}}$, and every z in Δ is contained in some R_{C_n}. Also the set $R_{C_{n+1}} - \bar{R}_{C_n}$ is entirely contained in Δ. If $f(z)$ is analytic in Δ, then the integral

$$\varphi_n(z) = \frac{1}{2\pi i} \int_{C_n} \frac{f(\zeta)}{\zeta - z} d\zeta$$

defines an analytic function in R_{C_n}. Since for every z in the latter union of domains, $f(\zeta)/(\zeta - z)$ is analytic for ζ in $R_{C_{n+1}} - \bar{R}_{C_n}$, we have

$$\varphi_n(z) = \varphi_{n+1}(z), \qquad z \varepsilon R_{C_n}$$

This defines an analytic function $\varphi(z) = \varphi_f(z)$ in $\tilde{\Delta}$ which is equal to $\varphi_n(z)$ in R_{C_n}. Also we may write

$$\varphi_f(z) = \frac{1}{2\pi i} \int_{C} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for $z \varepsilon R_C$, where C is any retrosection of Δ which is homologous to a complete retrosection.

We now turn to the proof of the theorem. Let $f(z; u_1 \cdots, u_n)$ be any function analytic in (2). For any fixed point $u \varepsilon U$ we introduce the function

$$\varphi_f(z) \equiv \varphi_f(z;\ u_1, \cdots, u_n)$$

as described above, and this leads to a function which is uniquely defined in $\tilde{D}_z$. We claim that it is analytic in all its variables. In fact if (c, z_0) is any given point of $\tilde{D}_z$, we take a retrosection C in $\Delta(c)$ such that $z_0 \varepsilon R_C$, and repeating the argument of section 1 we see that for all points in the interior of (7) our new function is given by

$$\varphi_f(z; u_1, \cdots, u_n) = \frac{1}{2\pi i} \int_C \frac{f(\zeta; u_1, \cdots, u_n)}{\zeta - z} d\zeta \tag{8}$$

Since, for the moment, C is a fixed curve, and the integrand in (8) is analytic in all variables occurring, the function (8) is analytic in the interior of (7), and in particular in the neighborhood of the arbitrary point (c, z_0).

Now, if for a *particular* point c all components of $\Delta(u)$ are simply connected, then, by Cauchy's formula, the function (8) is $f(z; u_1, \cdots, u_n)$ itself as stated in theorem 1, which completes the proof of that theorem.

Of importance is a special case which we will state separately with a sketch of a direct proof.

Theorem 2. *If corresponding to any $u \varepsilon U$, there exist two Jordan curves $C' = C'(u)$, $C'' = C''(u)$, with C' contained in $R_{C''}$, such that, for each u, the point set $\Delta(u)$ is contained in $R_{C''}$ but contains the whole annulus*

$$R_{C''} - \bar{R}_{C'} \tag{9}$$

if corresponding to any $c \varepsilon U$, there exists a neighborhood (5), *and a curve C, such that C is a complete retrosection of the annulus* (9) *for all u in* (5), *and if for a particular $c \varepsilon U$, there exists a neighborhood* (5) *such that for u in that neighborhood, $\Delta(u)$ contains, and thus is identical with $R_{C''}$, then $\tilde{D}_z$ is an analytic completion of D.*

If C is a curve as mentioned in the theorem, we set up the integral (8). It is an analytic function in (7). For fixed c it is extensible to an analytic function of z in all of $\tilde{\Delta}(c)$, and it is not hard to see, using the argument of the proof of Theorem 1, that it is analytic in all variables in

$$u \varepsilon U_{\rho,c}; \qquad z \varepsilon \tilde{\Delta}(u)$$

Thus we obtain an analytic function in $\tilde{D}_z$, and in the particular neighborhood of the particular point as stipulated in the theorem it coincides with $f(z; u_1, \cdots, u_n)$. This completes the proof.

Theorem 2 is a detailed wording of what is often called the "continuity theorem." Its extension to real variables u_m is due to Severi.

§3. CONVEX DOMAINS

We continue with Euclidean space E_{n+2} of the variables $(z; u_1, \cdots, u_n)$, $n \geq 1$.

If A is a bounded convex domain, in the ordinary definition of convexity, and B is another convex domain whose closure $\bar{B}$ is contained in A, then the point set $C = A - \bar{B}$ will be called a *convex shell*. We now project our point sets on the $(u_1, \cdots, u_n)$-space. The projection of A is a convex domain U_A, and the projection of C is the same domain. The projection B is convex subdomain U_B, and $U_A - U_{\bar{B}}$ is again a convex shell. However, for $n = 1$ this "convex shell" degenerates into two separated intervals on the u-axis.

Writing A and C in the form

$$A: \qquad [u \in U_A,\ z \in \Delta_A(u)]$$
$$C: \qquad [u \in U_A,\ z \in \Delta_C(u)]$$

we see that for $u \in U_A - U_{\bar{B}}$, both $\Delta_A(u)$ and $\Delta_C(u)$ are the same two-dimensional convex domains; whereas for $u \in U_{\bar{B}}$, $\Delta_C(u)$ arises from $\Delta_A(u)$ by taking out either a point, or a convex curve and its interior.

Now, let U be any domain which is part of U_A and which contains points of the shell $U_A - U_{\bar{B}}$, and such that the two point sets

$$D' = [u \in U,\ z \in \Delta_A(u)]$$
$$D = [u \in U,\ z \in \Delta_C(u)]$$

are domains in E_{n+2}. By Theorem 2, D' is an analytic completion $\tilde{D}_z$ of D.

Now a subdomain U of the given description will arise if we form the intersection of U_A with a half space $u_1 > a$, or $u_1 < b$, and for $n \geq 2$, we may take just any layer

$$a < u_1 < b \tag{10}$$

Thus we obtain

Theorem 3. *If a convex shell in $(z, u_1, \cdots, u_n)$ is given in the form*

$$[u \in U_0,\ z \in \Delta(u)] \tag{11}$$

and if U is the intersection of U_0 with any half space $u_1 > a$, or $u_1 < b$, or for $n \geq 2$, with any layer (10), *and if D is the domain*

$$[u \in U,\ z \in \Delta(u)] \tag{12}$$

then the domain

$$\tilde{D}_z: \qquad [u \in U,\ z \in \tilde{\Delta}(u)]$$

is an analytic completion of D.

So far we have singled out one real variable. However, if all variables occurring are complex a much smoother result emerges.

Theorem 4. *If $\tilde{R}$ is a convex domain in the space*

$$(z_1, \cdots, z_k),\ z_j = x_j + iy_j, \qquad k \geq 2$$

if R is a convex shell whose outer boundary is the boundary of $\tilde{R}$, and if $\tilde{D}$ and D are the intersections of $\tilde{R}$ and R with a spatial layer

$$a < \Sigma_{m=1}^{k}(\alpha_m x_m + \beta_m y_m) < b \tag{13}$$
$$(-\infty \leq a < b \leq \infty)$$

then $\tilde{D}$ is an analytic completion of D. In particular $\tilde{R}$ is an analytic completion of R.

In fact, we introduce any nonsingular linear transformation

$$z'_m = C_{m1}z_1 + \cdots + C_{mk}z_k \tag{14}$$

which is such that

$$C_{2m} = \alpha_m - i\beta_m$$

where α_m, β_m are the constants in (13). This linear transformation carries convex sets into convex sets and analytic functions into analytic functions, and the layer (13) into

$$a < \operatorname{Re}(z'_2) < b$$

Thus Theorem 4 follows from Theorem 3 if we replace z'_1 by z; x'_2, y'_2 by u_1, u_2; x'_3, y'_3 by u_3, u_4, etc.

For later use we draw a very special conclusion (and also a well-known one), namely

Theorem 5. *Let P be a boundary point of a hypersphere in $(z_1, \cdots, z_k)$-space $(k \geq 2)$ and let R be the portion of a neighborhood of P which lies outside the (closed) hypersphere. Then there exists a neighborhood N of P such that $N + R$ is an analytic completion of R.*

The proof follows easily from Theorem 2. Without loss in generality we will suppose that the hypersphere has the equation

$$|z_1|^2 + \cdots + |z_k|^2 = 1$$

and that P has the coordinates $(0, a_2, \cdots, a_k)$ with, of course,

$$|a_2|^2 + \cdots + |a_k|^2 = 1$$

Then by the nature of the domain R there exists a positive number h such that the domain

$$D: \quad |z_2 - a_2| < h, \qquad \cdots, \qquad |z_k - a_k| < h,$$
$$1 - |z_2|^2 - \cdots - |z_k|^2 < |z_1|^2 < h^2$$

lies in R. Now Theorem 2 applies at once with $\mathrm{Re}(z_2)$, $\mathrm{Im}(z_2)$, $\cdots$, $\mathrm{Re}(z_k)$, $\mathrm{Im}(z_k)$ playing the role of $u_1, \cdots, u_n$ and z_1 playing the role of z. Thus U is the domain

$$U: \qquad |z_j - a_j| < h, \qquad j = 2, \cdots, k$$

and $\Delta(z_2, \cdots, z_k)$ is the annulus

$$1 - |z_2|^2 - \cdots - |z_k|^2 < |z_1|^2 < h^2$$

For $(z_2, \cdots, z_k)$ in a suitably chosen subdomain of U we will have $|z_2|^2 + \cdots + |z_k|^2 > 1$ and hence for such points the domain $\Delta(z_2, \cdots, z_k)$ becomes the entire circle $|z_1| < h$. Hence by Theorem 2 the domain D has the domain $\tilde{D}_{z_1}$: $|z_2 - a_2| < h, \cdots, |z_k - a_k| < h$, $|z_1| < h$ as an analytic completion.

Now since $D \subset R$ it follows by Theorem 4 of Chapter II that $R + \tilde{D}_{z_1}$ is an analytic completion of R. This concludes the proof of Theorem 5.

§4. Modified Convexity

In the present section we will touch upon a topic which is closely connected with the so-called "analytic convexity" and "condition of Levi." The results offered will not be needed elsewhere and the proofs will be sketchy.

Theorem 6. *For $n \geq 2$, let $F(x, y; u_1, \cdots, u_n)$, $x + iy = z$, be a real continuous function in a neighborhood*

$$(15) \qquad |z| < \alpha, \qquad u_1^2 + \cdots + u_n^2 < \beta^2$$

of the origin and let it vanish at the origin, and let

$$(16) \qquad F(x, y; u_1, 0, 0, \cdots, 0) > 0$$

for

$$(17) \qquad |z|^2 + u_1^2 > 0$$

If a domain D contains the points for which $F > 0$, but none of the points for which $F \leq 0$, and if for every $\alpha' \leq \alpha$, $\beta' \leq \beta$, the neighborhood

$$|z| < \alpha', \qquad u_1^2 + \cdots + u_n^2 < \beta'^2$$

intersects D in a domain, then there exists a neighborhood N of the origin, such that $D + N$ is an analytic completion of D.

Proof. Letting $u_1 = 0$ in (16) we see that

$$F(x, y; 0, 0, \cdots, 0) > 0$$

for $0 < |z| < \alpha$, hence for some neighborhood

$$U: \qquad u_1^2 + \cdots + u_n^2 < \rho^2, \qquad \rho < \beta$$

and some annulus

$$\Delta: \qquad \alpha_2 < |z| < \alpha_1, \qquad \alpha_1 < \alpha$$

we have $F > 0$ for

$$u \in U, \qquad z \in \Delta \tag{18}$$

On the other hand, for $u_1 \neq 0$ we have $F(x, y; u_1, 0, \cdots, 0) > 0$ in the full circle $|z| < \alpha$, hence given $\alpha_1 < \alpha$ there exists a "small" cube

$$U_0: \qquad p_j < u_j < q_j, \qquad j = 1, \cdots, n$$

such that $F > 0$ for

$$u \in U_0, \qquad z \in \tilde{\Delta}$$

where $\tilde{\Delta}$ is $|z| < \alpha_1$. We now consider the neighborhood

$$N: \quad u \in U; \qquad z \in \tilde{\Delta} \tag{19}$$

By Theorem 2, it is the analytic completion of the intersection $D \cap N$, the latter intersection being a domain by assumption. Now by Theorem 4 of Chapter II, if D_1, D_2 are two domains with nonempty intersection D_1D_2 and if D_1' is any analytic completion of D_1, then $D_1' + D_2$ is an analytic completion of $D_1 + D_2$. Thus in the present case $D + N$ is an analytic completion of D.

It is very tempting to replace relations (16), (17) by the more sweeping relations

$$F(x, y; 0, \cdots, 0) > 0 \qquad \text{for} \qquad 0 < |z| < \alpha \tag{20}$$

That this cannot be done without restrictions is obvious from the example

$$F(x, y; u_1, u_2) = (x^2 + y^2) - (u_1^2 + u_2^2)$$

It satisfies (20), and yet the function $f(z; u_1, u_2) = w/z$, $w = u_1 + iu_2$, is analytic for $F > 0$, that is for $|w| < |z|$, and yet is not continuable into any full neighborhood including the origin $z = 0$, $w = 0$. The

reason for this failure is the vanishing of all partial derivatives of first order of the function F at the origin, as we shall see presently.

Theorem 7. *The conclusion of Theorem 6 also holds if, for $n \geq 1$, $F(x, y; u_1, \cdots, u_n)$ has the form*

$$u_1 + G(x, y; u_1, \cdots, u_n) \tag{21}$$

in a neighborhood (15), *if $G(x, y; u)$ has continuous partial derivatives of the first order, if these derivatives vanish at the origin, and if relation* (20) *holds.*

We put

$$\begin{aligned} H(x, y) &\equiv G(x, y; 0) \\ K(x, y; u) &\equiv G(x, y; u) - G(x, y; 0) \end{aligned}$$

Since $K(x, y; 0) = 0$, and since $K(x, y; u)$ has partial derivatives of the first order in $(x, y; u)$ which are uniformly continuous in $(x, y; u)$ and which vanish at the origin $x = y = u = 0$, there exists a neighborhood

$$|z| < \alpha', \qquad \Sigma_{m=1}^{n} u_m^2 < \beta'^2 \tag{22}$$

with $\alpha' < \alpha$, $\beta' < \beta$, such that for all points in this neighborhood

$$|K(x, y; u)| \leq \tfrac{1}{4}(u_1^2 + \cdots + u_n^2)^{\frac{1}{2}} \tag{23}$$

We will now exploit (20). In the first place we have $H(x, y) \geqslant 0$ everywhere. Also corresponding to any annulus

$$\alpha_2 < |z| < \alpha_1, \qquad \text{with} \qquad \alpha_1 < \alpha' \tag{24}$$

there exists a sufficiently small neighborhood

$$U: \qquad |u_m| < \rho, \qquad m = 1, \cdots, n \qquad (\sqrt{n}\,\rho < \beta')$$

such that for $u \in U$, and z in (24), we have

$$F(x, y; u) > 0 \tag{25}$$

Now, in the subdomain

$$U_0: \qquad +\sqrt{u_2^2 + \cdots + u_n^2} < u_1 < \rho, \qquad u_1 > 0$$

of U, we have

$$u_1 + H(x, y) + K(x, y; u) > u_1 - \tfrac{1}{2}u_1 + H(x, y) > 0 \tag{26}$$

where (x, y) can be any point of the complete circle $|z| < \alpha$. The balance of the proof is as for Theorem 6.

§5. Maximal Domains

We will repeatedly require a familiar principle which we will first formulate in axiomatic terms. We consider a set of elements T, U, V, $\cdots$ and we assume that certain pairs of elements are partially ordered by a relation $T \leq U$, with the properties (i) $T \leq T$, (ii) if $T \leq U$ and $U \leq T$ then $T = U$, and (iii) if $T \leq U$ and $U \leq V$, then $T \leq V$. If $\{T_\alpha\}$ is any subset, and U is any element, which may or may not belong to the subset, then U is called a maximum of the subset $\{T_\alpha\}$ if the relation $U < T_\alpha$ is false for all α. In particular, a maximum of the total set is an element which is not exceeded by any other element. A well-ordered sequence $T_1, T_2, \cdots, T_\omega, T_{\omega+1}, \cdots$ is called monotonely increasing if $\alpha < \beta$ implies $T_\alpha \leq T_\beta$.

The principle is as follows.

If every well-ordered monotonely increasing sequence in a set has a maximum, then the total set has a maximum.

In our applications, $\{T, U, V, \cdots\}$ will be a family of domains, and $T < U$ will denote that T is a proper subdomain of U.

§6. Radiated Domains

If $\zeta = (\zeta_1, \cdots, \zeta_k)$ are complex numbers with

$$\Sigma_{j=1}^{k}|\zeta_j|^2 = 1 \tag{27}$$

and if $0 < r < \infty$, then the numbers $(r; \zeta)$ are polar coordinates, in the space of the complex numbers $(z_1, \cdots, z_k)$, $z_j = r\zeta_j$, except for the origin. The points $r = 1$, as described by (27), compose the unit sphere Σ with center at the origin. For fixed ζ and for

$$0 \leq \rho(\zeta) < \sigma(\zeta) \leq \infty \tag{28}$$

the relation

$$\rho(\zeta) < r < \sigma(\zeta) \tag{29}$$

describes an open connected set on that ray through the origin which is determined by ζ. If B is any point set of the surface Σ, and if $\rho(\zeta)$, $\sigma(\zeta)$ are defined throughout B, then the point set

$$\zeta \in B, \qquad \rho(\zeta) < r < \sigma(\zeta) \tag{30}$$

in $(z_1, \cdots, z_k)$-space will be called a *radiated* point set. We are exclusively interested in *radiated domains*. If (30) is a domain D, then B is a domain in Σ, and $\rho(\zeta)$ and $\sigma(\zeta)$ have an important property. If ζ^0 is a point of B, and if $\rho_0 \leq r \leq \sigma_0$ is a closed subset of $\rho(\zeta^0) <$

$r < \sigma(\zeta^0)$ then D contains a $2k$-dimensional neighborhood of the segment. Hence there exists a neighborhood B_0 of ζ^0 such that for $\zeta \varepsilon B_0$, $\rho(\zeta) \leq \rho_0$, $\sigma_0 \leq \sigma(\zeta)$. In other words, *the function* $\rho(\zeta)$ *is upper semicontinuous and the function* $\sigma(\zeta)$ *is lower semicontinuous.* Since $\rho(\zeta^0) < \sigma(\zeta^0)$, we have in particular the following conclusion. There exist a neighborhood B_1 of ζ^0 and an $\epsilon > 0$ such that $\sigma(\zeta) > \rho(\zeta^0) + \epsilon$ for $\zeta \varepsilon B_1$.

Theorem 8. *Let* $k \geq 2$. *If a domain* D *has the form* (30), *and if for a point* ζ^0 *of* B, $\rho(\zeta)$ *has a local maximum* $\rho = \rho(\zeta^0) > 0$, *then there exists a function*

$$\tilde{\rho}(\zeta) \leq \rho(\zeta), \qquad \tilde{\rho}(\zeta^0) < \rho \tag{31}$$

such that the point set

$$\zeta \varepsilon B, \qquad \tilde{\rho}(\zeta) < r < \sigma(\zeta) \tag{32}$$

is a domain $\tilde{D}$ *which is an analytic completion of* D.

Proof. By assumption there exists a neighborhood B_2 of ζ^0, such that $\rho(\zeta) \leq \rho(\zeta^0) = \rho$ for ζ in B_2. Combining this with the conclusion stated before, we see that there exist a neighborhood B_3 of ζ^0 and an $\epsilon > 0$ such that the domain

$$D_0\colon \quad \zeta \varepsilon B_3, \qquad \rho < r < \rho + \epsilon$$

is part of D. The boundary of D_0 includes a fragment of the sphere $|z_1|^2 + \cdots + |z_k|^2 = \rho^2$, and the point $z_j^0 = \rho\zeta_j^0$ is an inner point of the fragment. Hence, by Theorem 5, there exists a neighborhood N of z^0 such that a function which is analytic in D_0 is analytic in $D_0 + N$. Now N contains a subneighborhood of the form

$$N_0\colon \qquad \zeta \varepsilon B_4, \qquad \rho - \delta < r < \rho + \delta$$

where B_4 is a sufficiently small neighborhood of ζ^0, ($B_4 \subset B_3$), and $0 < \delta < \epsilon$.

We next claim that the intersection R of the neighborhood N_0 and the entire domain D is itself a domain. In fact, R is composed of the points $[\zeta \varepsilon B_4, \max(\rho(\zeta), \rho - \delta) < r < \rho + \delta]$. If (ζ', r') and (ζ^2, r^2) are any two of its points, then the first can be connected within R with the point $(\zeta', r = \rho + \delta/2)$, and the second with the point $(\zeta^2, r = \rho + \delta/2)$. The new points are both located in the point set

$$\zeta \varepsilon B_4, \qquad \rho < r < \rho + \delta$$

which is a subdomain of R, and are thus connectible within R.

Now the domain $\tilde{D} = D + N_0$ has the form (32) with

$$\tilde{\rho}(\zeta) = \begin{cases} \min(\rho(\zeta), \rho - \delta) & \text{for} \quad \zeta \in B_4 \\ \rho(\zeta) & \text{for} \quad \zeta \in B - B_4 \end{cases}$$

and as we have seen $D_0 + N$ is an analytic completion of D_0. Since $D_0 \subset D$ and $N_0 \subset N$ this means by Theorem 4 of Chapter II that $D + N_0$ is an analytic completion of D.

Theorem 9. *Let $k \geq 2$. If a domain D has the form* (30) *then there exists a function $\tilde{\rho}(\zeta)$, with $0 \leq \tilde{\rho}(\zeta) \leq \rho(\zeta)$, such that $\tilde{\rho}(\zeta)$ has no local maxima > 0, and such that the point set*

$$\zeta \in B, \qquad \tilde{\rho}(\zeta) < r < \sigma(\zeta)$$

is a domain $\tilde{D}$ which is an analytic completion of D.

In particular, if there exists a subdomain B_0 of B, whose closure $\bar{B}_0$ is contained in B and if for $\zeta \in B - B_0$, $\rho(\zeta) = 0$, then an analytic completion $\tilde{D}$ of D consists of the conical domain $\zeta \in B$, $0 < r < \sigma(\zeta)$.

Proof. We introduce the family of domains of the form

$$\zeta \in B, \qquad \bar{\rho}(\zeta) < r < \sigma(\zeta)$$

with $0 \leq \bar{\rho}(\zeta) \leq \rho(\zeta)$ which, if larger than D, are analytic completions of D. The union of any well ordered monotonely increasing sequence of such domains is again a domain of this form, and any function analytic in D has an analytic continuation in the union. Also the union is a maximum for the elements of the sequence. Hence by the principle enunciated in section 5, our total family has a maximum. By Theorem 8, the function $\tilde{\rho}(\zeta)$ belonging to the maximal domain may have no local maxima > 0.

As for the second half of the theorem, our function $\tilde{\rho}(\zeta)$ is upper semicontinuous in $\bar{B}_0$ and ≥ 0 in $\bar{B}_0$. Now an upper semicontinuous function on a compact set must have an absolute maximum. But by what we have just proved this maximum cannot be > 0. Hence the absolute maximum must be 0 and this completes the proof of the theorem.

Maximal radiated domains may be formed in various ways. We shall indicate two particular ways. One way is to form it as we did in the proof just given, working only from below, i.e. working only with $\rho(\zeta)$. Such a maximal domain we will call *maximal from below.* The second maximal domain which we discuss we form as follows. With a radiated domain D as in (30) we introduce the family of domains of the form

$$\zeta \in B, \qquad \bar{\rho}(\zeta) < r < \bar{\sigma}(\zeta)$$

with $0 \leq \bar{\rho}(\zeta) \leq \rho(\zeta)$, $\sigma(\zeta) \leq \bar{\sigma}(\zeta) \leq \infty$ which, if larger than D, are analytic completions of D. This family again has a maximum, which is itself a radiated domain. We will call this domain *maximal from below and above.*

As a corollary to Theorem 8 we have:

Corollary 1. *Let* $k \geq 2$. *If* (30) *is a radiated domain, maximal either from below, or from below and above, then* $\rho(\zeta)$ *has no local maxima* > 0. *If, in addition, the basis* B *in* (30) *is the whole unit sphere* (27), *then* $\rho(\zeta) \equiv 0$. *In this case it follows that* (30) *has as an analytic completion the domain* $[\zeta \varepsilon B, 0 \leq r < \sigma(\zeta)]$.

The last statement follows from Theorem 1, say. It uses the fact that $\sigma(\zeta)$ is a lower semicontinuous function everywhere > 0 on the unit sphere. As such it must have a minimum $m > 0$ and thus our radiated domain (30) contains a domain of the form $0 < |z_1|^2 + \cdots + |z_k|^2 < m^2$. But this latter domain has the analytic completion $0 \leq |z_1|^2 + \cdots + |z_k|^2 < m^2$. This yields the last statement of the corollary.

§7. Domains of Circular Type

We consider the space E_{2k} of complex variables $z = (z_1, \cdots, z_k)$. We add a complex parameter t, and we introduce nonnegative integer exponents $\alpha_1, \cdots, \alpha_k$, not all 0. If z is any fixed point, then the point set

$$(33) \qquad (t^{\alpha_1}z_1, \cdots, t^{\alpha_k}z_k), \qquad |t| = 1$$

is a topological image of a circle. We will term it an *orbit*, and we will say that it is generated by the point z. It is not hard to see that an orbit is generated by any of its points. We will say that a domain A, (or a general point set), in E_{2k} is of circular type, if it consists entirely of orbits.

In the case $\alpha_1 = \cdots = \alpha_k = 1$, an orbit is a point set

$$(34) \qquad (e^{i\vartheta}z_1, \cdots, e^{i\vartheta}z_k), \qquad 0 \leq \vartheta < 2\pi$$

and the domain is called *circular* (Carathéodory). They arise naturally as domains of continuous convergence for series of homogenous polynomials in $(z_1, \cdots, z_k)$.

In the case $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_k = 0$ an orbit is

$$(35) \qquad (e^{i\vartheta}z_1, z_2, \cdots, z_k), \qquad 0 \leq \vartheta < 2\pi$$

and the domain is called a *Hartogs domain.* It arises naturally as the domain of continuous convergence for a series

$$(36) \qquad \Sigma_{n=0}^{\infty} a_n(z_2, \cdots, z_k)z_1^n$$

In what follows, any variable z_j for which $\alpha_j = 0$ may also be assumed to be real since it is not affected by the requirement of circularity. Furthermore in Theorem 10 to follow, instead of requiring that the domain A contain the origin $z_1 = 0, \cdots, z_k = 0$ it is sufficient to require that A shall contain a point $(z_1^0, \cdots, z_k^0)$ for which $z_j^0 = 0$ whenever $\alpha_j \neq 0$, since the other components can be made 0 by translations which do not affect the circularity. We finally note that combinations of arbitrary integers α_j as exponents have been first introduced by H. Cartan.

We now associate with each domain A certain completion $\tilde{A}$. If A is the union of orbits, then $\tilde{A}$ is defined to be the unions of the corresponding "disks"

$$(37) \qquad (t^{\alpha_1}z_1, \cdots, t^{\alpha_k}z_k), \qquad 0 < |t| \leq 1$$

and we now can state a theorem.

Theorem 10. *The completion $\tilde{A}$ of A is again a domain, and if A contains the origin $z_1 = \cdots = z_k = 0$, then $\tilde{A}$ is an analytic completion of A.*

Proof. The proof is fundamentally very simple although the details will be somewhat elaborate. An orbit is a compact set. Hence if A contains the orbit (33) it also contains an enclosing "channel"

$$(38) \qquad (t^{\alpha_1}z_1, \cdots, t^{\alpha_k}z_k), \qquad \rho < |t| < \sigma, \qquad \rho < 1 < \sigma$$

We now associate with each $z \varepsilon A$ the largest available channel $\rho(z) < |t| < \sigma(z)$ in A, and we now consider in the space of the variables $(z_1, \cdots, z_k, t)$ the point set

$$(39) \qquad z \varepsilon A; \qquad \rho(z) < |t| < \sigma(z)$$

where

$$0 \leq \rho(z) < 1 < \sigma(z) \leq \infty$$

This point set in E_{2k+2} will be denoted by D, and it is not hard to see that D is a domain.

For some points z in A we may have not only an enclosing channel (38) but the entire point set

$$(t^{\alpha} z_1, \cdots, t^{\alpha_k}z_k), \ |t| < \sigma(z)$$

within A. This will be certainly the case for the origin, and for a neighborhood of the origin, whenever A contains the origin. Incidentally for the origin we have of course $\sigma(0) = \infty$. Thus we introduce in E_{2k+2} another point set H, which includes D, and which is

again a domain, namely

$$H: \qquad z \varepsilon A; \qquad t \varepsilon \Delta(z)$$

where $\Delta(z)$ is either the previous annulus $\rho(z) < |t| < \sigma(z)$; or the entire circle $|t| < \sigma(z)$ whenever possible. We further introduce two completions,

$$\tilde{D}: \qquad z \varepsilon A; \qquad 0 < |t| < \sigma(z)$$
$$\tilde{H}: \qquad z \varepsilon A; \qquad |t| < \sigma(z)$$

We now consider the transformation of variables

$$\zeta_j = t^{\alpha_j} z_j, \qquad j = 1, \cdots, k \tag{40}$$
$$\tau = t$$

It is a one-to-one analytic transformation of the domain

$$z_1, \cdots, z_k \text{ arbitrary}; \qquad t \neq 0 \tag{41}$$

into the domain

$$\zeta_1, \cdots, \zeta_k \text{ arbitrary}; \qquad \tau \neq 0 \tag{42}$$

We denote the images of D and $\tilde{D}$ in (42) by M and $\tilde{M}$ respectively. They are again domains. Lastly, we project M and $\tilde{M}$ on the coordinate space $(\zeta_1, \cdots, \zeta_k)$, that is on the manifold $\tau = 0$, and the reader will verify that these projections are nothing else but the original point sets A and $\tilde{A}$, only now in terms of $(\zeta_1, \cdots, \zeta_k)$ instead of $(z_1, \cdots, z_k)$. Since the projection of a domain on a coordinate "plane" is again a domain, we first conclude that $\tilde{A}$ is a domain.

We now assume that A contains the origin. If $f(\zeta_1, \cdots, \zeta_k)$ is analytic in A then the function

$$g(z_1, \cdots, z_k; t) \equiv f(t^{\alpha_1} z_1, \cdots, t^{\alpha_k} z_k) \tag{43}$$

is analytic in H. By Theorem 2, (43) has a continuation in $\tilde{H}$, and in particular in $\tilde{D}$. Transcribing the continued function into the variables (40), we obtain a function

$$F(\zeta_1, \cdots, \zeta_k; \tau) \tag{44}$$

in the domain $\tilde{M}$. And it is not hard to see that in the subdomain M the latter function is the original function $f(\zeta_1, \cdots, \zeta_k)$. Thus the partial derivative of (44) with respect to τ vanishes in M, and being analytic it vanishes in all $\tilde{M}$. Thus F is independent of τ throughout $\tilde{M}$, and thus it gives rise to an analytic function in the projection

$\tilde{A}$ of $\tilde{M}$, and this function is an analytic continuation of $f(\zeta)$ from A to $\tilde{A}$.

§8. Domains of Multi-circular Type

We take m parameters $t_1, \cdots, t_m$, and with nonnegative integer exponents $\alpha_{j\mu}$ we form monomials

$$\beta_j(t) = t_1^{\alpha_{j1}} \cdots t_m^{\alpha_{jm}}, \qquad j = 1, \cdots, k$$

We now replace definition (33) of an orbit by

$$(\beta_1(t)z_1, \cdots, \beta_k(t)z_k), \qquad |t_1| = |t_2| = \cdots = |t_m| = 1$$

and for definition of a "disk" we have the inequalities

$$0 < |t_1| \leq 1, \qquad 0 \leq |t_2| \leq 1, \qquad \cdots, \qquad 0 < |t_m| \leq 1$$

Theorem 10 also holds for multi-circular domains, as can be proven by induction on m. In fact, putting

$$\beta_j(t) = t_1^{\alpha_{j1}}\gamma_j(t)$$

we have $\beta_j(t) = t_1^{\alpha_{j1}}$ for the special values $t_2 = \cdots = t_m = 1$. Thus, by Theorem 10, the domain $\tilde{A}_1$ consisting of partial disks

$$(t_1^{\alpha_{11}}z_1, \cdots, t_1^{\alpha_{k1}}z_k), \qquad 0 < |t_1| \leq 1$$

is an analytic completion of A. On the other hand, for $t_1 = 1$, $\beta_j(t) = \gamma_j(t)$, hence the manifolds

$$(\gamma_1(t)z_1, \cdots, \gamma_k(t)z_k), \qquad |t_2| = \cdots = |t_m| = 1$$

belong to A. Thus, by what we have just proven, $\tilde{A}_1$ includes the partial disks

$$\begin{gathered}(\beta_1(t)z_1, \cdots, \beta_k(t)z_k \\ 0 < |t_1| \leq 1, \qquad |t_2| = \cdots = |t_m| = 1\end{gathered}$$

Thus, assuming that our theorem for multi-circular regions has already been proved for $m - 1$, we now obtain analytic continuation into all of $\tilde{A}$.

The most important case of a multi-circular domain is $m = k$, $\beta_j(t) = t_j$. The orbits are

$$\begin{gathered}(e^{i\vartheta_1}z_1, \cdots, e^{i\vartheta_k}z_k) \\ 0 \leq \vartheta_j < 2\pi, \qquad j = 1, \cdots, k\end{gathered} \tag{45}$$

and the domain is called a *Reinhardt domain*. It arises naturally as a domain of absolute convergence for a power series in several variables.

If a Reinhardt domain D contains the origin then the preceding results show that D has as its analytic completion the domain $\tilde{D}$ which contains all points $(t_1z_1, \cdots, t_kz_k)$ with $|t_1| \leq 1, \cdots, |t_k| \leq 1$ and z in D. The domain $\tilde{D}$ is called the Reinhardt analytic completion of D.

§9. SOME CONCLUDING REMARKS

Throughout the present chapter we have always considered the case in which there occurs at least one *complex* variable, and one or more other variables (real or complex). The presence of this complex variable enabled us to apply Cauchy's integral formula for one complex variable, which forms the basis for this chapter. The presence of this complex variable is, however, more than a mere convenience, as easily constructed examples show. For example, by Theorem 1, say, the domain

$$D: \qquad 0 < |z|^2 + u_1^2 + \cdots + u_n^2 < 1, \qquad (z = x + iy)$$

has the analytic completion

$$\tilde{D}: \qquad |z|^2 + u_1^2 + \cdots + u_n^2 < 1$$

that is, every function $f(z; u_1, \cdots, u_n)$ analytic in D has an analytic continuation into $\tilde{D}$. But the function

$$\frac{1}{x^2 + y^2 + u_1^2 + \cdots + u_n^2}$$

is an analytic function of all of its *real* variables in D and yet it has no analytic continuation into $\tilde{D}$. Thus a domain D may have an analytic completion for analytic functions of the variables $(z; u_1, \cdots, u_n)$ and at the same time not for analytic functions of the variables $(x, y; u_1, \cdots, u_n)$. Of course, the converse of this is *not* true, for as we have seen in section 3 of Chapter II, if a given function $g(x, y, u_1, \cdots, u_n)$ analytic in a domain R has an analytic continuation into some domain $\tilde{R}$, and if before continuation it involves the variables x, y only in the combination $x + iy$, then this situation prevails after continuation.

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CHAPTER V

Singularities at Boundary Points

In the present chapter all variables will be complex and all functions analytic, unless otherwise stated.

§1. UNBOUNDED FUNCTIONS

Given a domain D, a point P on the boundary will be said to have the *frontier property* if corresponding to every compact set S in D and every $\epsilon > 0$, there exist a function $g(z)$ in D and a point Q in D outside S, both depending upon S and ϵ, such that $|Q - P| < \epsilon$, and $|g(z)| \leq 1$ for $z \in S$ but $|g(Q)| > 1$.

A function $f(z)$ in D is said to be *unbounded* at a boundary point P if there exists a sequence of points Q_n in D such that $Q_n \to P$ and $|f(Q_n)| \to \infty$. Obviously if there exists a function in D which is unbounded at P, then P has the frontier property. However, the converse is also true, and a good deal more.

Theorem 1. *Given a domain D, if $\{P\}$ is a set of points on the boundary each having the frontier property, then there exists a function $f(z)$ in D which is unbounded at every point of $\{P\}$.*

Proof. There exists a dense subset of $\{P\}$ which is at most countable, and if a function is unbounded at the points of the dense subset it is obviously also unbounded at the other points of $\{P\}$. Thus we may assume that the set $\{P\}$ is at most countable. We now form a sequence $P_1, P_2, P_3, \cdots$ with the following properties: each point P_n is an element of $\{P\}$ and each element of $\{P\}$ occurs in the sequence infinitely often. And we will construct a function $f(z)$ and a sequence of points $Q_1, Q_2, \cdots$, such that $|Q_n - P_n| \to 0$ and $|f(Q_n)| \to \infty$. This will suffice. In fact, every point of $\{P\}$ occurs in our sequence infinitely often, and thus, given P, there will exist a sub-sequence $\{Q'_n\}$ of $\{Q_n\}$ such that $Q'_n \to P$ and $|f(Q'_n)| \to \infty$.

We start from any sequence of compact sets $R_1, \cdots, R_p, \cdots$ in D such that $R_p \subset R_{p+1}$ and $\Sigma_{p=1}^{\infty} R_p = D$. By induction on n, $n \geq 1$, we will select a sequence of integers $p_1 < p_2 < \cdots < p_n < \cdots$, a sequence of points Q_n and a sequence of functions $f_n(z)$ in D such that, putting $S_n = R_{p_n}$ the following properties hold:

(i) $$S_n \subset S_{n+1}, \qquad \Sigma S_n = D$$

(ii) $$Q_n \in S_{n+1} - S_n$$

(iii) $$|Q_n - P_n| \leq \frac{1}{n}$$

(iv) $$|f_n(z)| \leq 1 \quad \text{for} \quad z \in S_n, \quad \text{but} \quad |f_n(Q_n)| > 1$$

We first put $S_1 = R_1$ and, since P_1 has the frontier property, we may choose $f_1(z)$ and Q_1 in $D - S_1$ so as to satisfy (iii) and (iv) for $n = 1$. If everything has been chosen suitably for $n \leq m$, we choose p_{m+1} large enough so that $S_{m+1} \equiv R_{p_{m+1}}$ shall include S_m and also the point Q_m (which is exterior to S_m). Since P_{m+1} has the frontier property there exist a function $f_{m+1}(z)$ and a point Q_{m+1} in $D - S_{m+1}$ which satisfy (iii) and (iv) for $n = m + 1$.

Finally, since $|f_n(Q_n)| > 1$ we can select a sequence of positive integers l_n, starting with $l_1 = 1$, and using induction on n, such that

(1) $$\frac{|f_n(Q_n)|^{l_n}}{n^2} - \Sigma_{m=1}^{n-1} \frac{|f_m(Q_n)|^{l_m}}{m^2} \geq n, \qquad (n \geq 2)$$

We now define our function $f(z)$ as the series

$$f(z) = \Sigma_{m=1}^{\infty} \frac{[f_m(z)]^{l_m}}{m^2}$$

Since $|f_m(z)| \leq 1$ for $z \in S_n$, $m \geq n$, the series is uniformly convergent in each S_n, and thus defines an analytic function in D since (as we may assume) each compact set in D is contained in some initial set R_p. Furthermore

$$|f(Q_n)| \geq \frac{|f_n(Q_n)|^{l_n}}{n^2} - \Sigma_{m=1}^{n-1} \frac{|f_m(Q_n)|^{l_m}}{m^2} - \Sigma_{m=n+1}^{\infty} \frac{1}{m^2}$$

and thus, by (1), $|f(Q_n)| \to \infty$. This completes the proof of the theorem.

A domain D is called a *domain of regularity* if there exists a function $f(z)$ in D which cannot be continued beyond D. We know by Theorem 1 that if every boundary point of a domain D has the frontier property then D is a domain of regularity.

Theorem 2. *Every boundary point of a convex domain D has the frontier property and thus D is a domain of regularity.*

For the sphere

(2) $$|z_1|^2 + \cdots + |z_k|^2 < 1$$

the proof is immediate. If $(\zeta_1, \cdots, \zeta_k)$ is a boundary point, the

function $(1 - \bar{\zeta}_1 z_1 - \cdots - \bar{\zeta}_k z_k)^{-1}$ is analytic in (2) but unbounded at $(\zeta_1, \cdots, \zeta_k)$.

In the general case we know by the theory of convex domains that through every boundary point P_0: $(z_1^0, \cdots, z_k^0)$ there passes a supporting plane

$$\Sigma_{j=1}^k [a_j(x_j - x_j^0) + b_j(y_j - y_j^0)] = 0 \qquad (z_j^0 = x_j^0 + iy_j^0)$$

which cuts no interior points of D and which has the property that D lies entirely on one side of it. With these a's and b's define

$$\alpha_{1p} = a_p - ib_p, \qquad p = 1, \cdots, k$$

and let $(\alpha_{2p}, \cdots, \alpha_{kp})$, $p = 1, \cdots, k$ be a set of complex constants such that the determinant $|\alpha_{jp}|$, $j, p = 1, \cdots, k$, is different from zero. Such a set obviously exists since not all the α_{1p} are zero. With these α's we form the nonsingular linear transformation

$$z_j' = \Sigma_{p=1}^k \alpha_{jp}(z_p - z_p^0), \qquad j = 1, \cdots, k$$

It carries the above supporting plane into $x_1' = 0$, the point P_0 into the origin and it places the domain D entirely in the half-space $x_1' > 0$ (or in $x_1' < 0$ as the case may be). Obviously the function $(z_1')^{-1}$ is analytic in D but unbounded at P_0.

§2. An Analytic Criterion for Completion

In the present section it will be advantageous to consider polycylinders (products of circles) whose several radii are all equal. If $r > 0$ is a number, not a vector, and $(z_1^0, \cdots, z_k^0)$ is any point then the polycylinder

(3) $$|z_j - z_j^0| < r, \qquad j = 1, \cdots, k$$

will be denoted by $C(z^0, r)$, and if S is any set in $(z_1, \cdots, z_k)$-space, then S^r will denote the union of all $C(z^0, r)$ for $z^0 \varepsilon S$. Thus S^r is a neighborhood of S of "radius" r.

Theorem 3. *Given a domain D, if S is a compact set in D, if, for some $r > 0$, the closure of S^r is likewise contained in D, if Q: $(z_1^0, \cdots, z_k^0)$ is a point in $D - S^r$, and if for every function $g(z)$ in D the relation*

(4) $$|g(Q)| \leq \max_{z \varepsilon S} |g(z)|$$

holds, then for every $f(z)$ in D the power-series

(5) $$f(z) = \Sigma a_{n_1 \cdots n_k}(z_1 - z_1^0)^{n_1} \cdots (z_k - z_k^0)^{n_k}$$

about the point Q is absolutely convergent in (3).

Proof. For any $\zeta \varepsilon D$ we have an expansion

$$f(z) = \Sigma a_{n_1 \cdots n_k}(\zeta)(z_1 - \zeta_1)^{n_1} \cdots (z_k - \zeta_k)^{n_k} \tag{6}$$

where

$$n_1! \cdots n_k! a_{n_1 \cdots n_k}(\zeta) = \left.\frac{\partial^{n_1 + \cdots + n_k} f(z)}{\partial z_1^{n_1} \cdots \partial z_k^{n_k}}\right]^{z=\zeta}$$

Thus $a_{n_1 \cdots n_k}(\zeta)$ is analytic in ζ over D. Now, by formula (14), Chapter II, if $|f(z)| \leq M$ in the closure of $C(\zeta, r)$ then

$$|a_{n_1 \cdots n_k}(\zeta)| \leq M r^{n_1 + \cdots + n_k} \tag{7}$$

Now since $f(z)$ is analytic in D, and since the closure of S^r lies in D, there exists a constant M such that $|f(z)| \leq M$ in the closure of S^r. Thus (7) holds for all ζ in S. We now apply the decisive assumption (4) to all functions $a_{n_1 \cdots n_k}(\zeta)$. As a consequence, (7) holds also for $\zeta_j = z_j^0$ and thus (5) converges in $C(z^0, r)$, and this completes the proof of the theorem.

The criteria for existence of an analytic completion of a domain D, as derived in Chapter IV, bore on the geometric structure of D. The criterion to follow is seemingly not so.

We will say that *D is locally connected at the boundary point P*, if every spatial neighborhood of P contains a subneighborhood whose intersection with D is connected.

Theorem 4. *If D has the frontier property at P then there does not exist an analytic completion D' of D which contains the point P.*

However, if D does not have the frontier property at P, and if D is locally connected at P, then such a completion does exist.

The first part of the theorem is a conclusion, already drawn, from Theorem 1.

Now for the converse. If P does not have the frontier property, then there exist a compact set S in D and a spherical neighborhood N_0 of P, with closure outside S and such that for every $f(z)$ in D and every Q in $N_0 \cdot D$,

$$|f(Q)| \leq \max_{z \varepsilon S} |f(z)|$$

By Theorem 3, there exists an $r > 0$ such that for every $Q = Q(z^0)$ the series (5) converges in $C(Q; r)$. Now choose a subneighborhood N of P contained in N_0 and of diameter $< \dfrac{r}{4}$ whose intersection with D is connected, and choose any fixed point Q in this intersection.

Then (5) is the continuation of $f(z)$ from D into $D' \equiv D + N$. The proof is finished.

Theorem 4 is very strong. For instance take the following conclusion:

Theorem 5. *If D is locally connected at P, and if every function $f(z)$ in D can be continued into some neighborhood N_f of P, then all functions in D can be continued into a common neighborhood N of P.*

In fact, by hypothesis, P cannot have the frontier property; if it had it, there would exist by Theorem 1 a function in D which is unbounded at P and hence not continuable beyond P. However, if P does not have the frontier property, D can be completed beyond P by Theorem 4.

Theorem 6. *If $\{P\}$ is a set of points on the boundary of a domain D at each point of which D is locally connected and if there is no completion of D containing any one point P, then there exists a function $f(z)$ in D which is unbounded at all points P.*

The theorem follows again from Theorems 1 and 4. A more general situation arises if we consider a radiated domain as in section 6, Chapter IV. Such a domain consists of "segments"

$$z = (r\zeta_1, \cdots, r\zeta_k), \qquad 0 \leq \rho(\zeta) < r < \sigma(\zeta) \leq \infty$$

where $\zeta = (\zeta_1, \cdots, \zeta_k)$ runs over a domain B on the unit sphere $|\zeta_1|^2 + \cdots + |\zeta_k|^2 = 1$. By an argument which was used in the course of section 6, Chapter IV it is easy to see that any top point of a segment, that is $r = \sigma(\zeta)(< \infty)$, is a boundary point at which D is locally connected, and so is every bottom point $r = \rho(\zeta)$, provided it is not the origin, that is, provided $\rho(\zeta) > 0$. Now assume that our radiated domain is maximal from below and above, in accordance with the definition given near the end of section 6, Chapter IV. Then by Theorem 6 there exists a function $f(z)$ in D which is unbounded at all those points (that is at the top and bottom points of all the segments, excluding those for which $\rho(\zeta) = 0$ or $\sigma(\zeta) = \infty$). Assume further that the basis of our radiated domain consists of the whole unit sphere. If D is maximal from below and above (or even only from below), then by Corollary 1 of section 6, Chapter IV, it follows that $\rho(\zeta) \equiv 0$ and every function in D is automatically analytic at the origin. Thus we may add the origin to our domain D. Defining a *star* domain to be a domain of the form [ζ contained on the whole unit sphere, $0 \leq r < \sigma(\zeta)$] and using Theorem 6 we now have the following generalization of Theorem 2.

Theorem 7. *A star domain is maximal if and only if it is a domain of regularity.*

§3. Relative Completion

We consider in a domain D a family of functions which is closed with respect to operations of addition, multiplication, multiplication by arbitrary constants, continuous convergence, and partial derivation. Since only these operations were used thus far, we may readily introduce the concept of completing D to a domain D' relative to the given family of functions, and our theorems (except for Theorem 5) will hold. Of course, the smaller the family of functions the better the chances for completion. However, there are some striking cases, when shrinkage does not necessarily affect the completion and we are going to elaborate one such case.

But before doing this we will insert a remark on functions of real variables. If D is a domain in real variables, and a family of real analytic functions is closed as before, then our theorems remain true provided the family is "locally compact." By this we mean that any set of functions of our family which are bounded on every compact subset S of D is a majorized set in S. This remark has some interest in connection with the fundamental Theorem 3, since for the validity of this theorem the family need not be closed under multiplication but only under partial derivation. It can be shown, for instance, that Theorem 3 can be adapted in its wording to apply to the solutions of the equation

$$\frac{\partial^2 f}{\partial z_1^2} + \cdots + \frac{\partial^2 f}{\partial z_k^2} = 0$$

(all z_j being real). However, in other circumstances, Theorem 1 has the advantage over Theorem 3. For instance, consider a domain D in a complex analytic coordinate space Σ_{2k}, see section 4, Chapter III. In this case, Theorem 1 carries over immediately, but Theorem 3 not at all so, since partial derivatives of a function are not again scalar functions but components of tensors. The way of remedying this situation would be to introduce a Riemannian metric on the space, but this would lead us too far afield. The metric can be avoided as long as the space Σ_{2k} is a "multi-valued" domain over the ordinary E_{2k}. However, any inclusion of points at "infinity" as interior points would necessitate the introduction of a metric.

§4. CONVEX TUBES

A *tube* T in the space of $z_j = x_j + iy_j$, $j = 1, \cdots, k$, is any point set which can be represented in the form

$$(\text{Re } z_1, \cdots, \text{Re } z_k) \subset S, \qquad -\infty < \text{Im} z_j < \infty$$

where S is any set in the k-dimensional space $(x_1, \cdots, x_k)$. If corresponding to any real point $(x_1^0, \cdots, x_k^0)$ we introduce the k-dimensional manifold

$$z_j = x_j^0 + iy_j, \qquad -\infty < y_j < \infty \tag{8}$$

then whenever T contains one point of (8) it contains all of them. The point set S in E_k will be called the *basis* of T, and we will also use the symbols T_S and S_T to denote respectively the tube with basis S and the basis of the tube T. Obviously T is open, closed, connected, convex in E_{2k} whenever S is so in E_k, and the convex hull of a tube is the tube over the convex hull of the basis. *Unless otherwise stated the tube T will be assumed to be a domain.* It is easy to see that the transformation

$$w_1 = e^{z_1}, \qquad \cdots, \qquad w_k = e^{z_k} \tag{9}$$

turns T into a Reinhardt domain D in $(w_1, \cdots, w_k)$-space which has the restrictive feature that all components of all its points are $\neq 0$; any manifold (8) turns into an orbit of D (see section 7, Chapter IV). The transformation is many to one, being periodic with period $2\pi i$ in each variable. Conversely if D is any Reinhardt domain which does not intersect any "plane" $w_j = 0$, then (9) transforms its universal covering space into a tube. Every analytic function in D goes into a multi-periodic function with periods $2\pi i$ in T, and vice versa.

The tube whose basis is a rectangle

$$R: \qquad \alpha_j \leq x_j \leq \beta_j, \qquad j = 1, \cdots, k$$

corresponds to the product of annuli

$$e^{\alpha_j} \leq |w_j| \leq e^{\beta_j} \tag{10}$$

If (10) lies in D, and if $F(w)$ is analytic in D, then we have Cauchy's formula, see formula (9), Chapter II, with each curve C_j being the sum of the directed boundary circles of the annulus (10). Hence we can derive a Laurent expansion

$$F(w) = \Sigma_{n_j=-\infty}^{\infty} a_{n_1 \cdots n_k} w_1^{n_1} \cdots w_k^{n_k} \tag{11}$$

in analogy to the classical derivation for the case $k = 1$, in much the same way as we derived in Chapter II the multiple power-series for products of circles. The Laurent series is absolutely convergent in the interior of the annulus, and, it is easily seen by induction on k that it is uniquely determined, no matter how arrived at. In particular, if $F(w)$ can be continued into the product of the circles $|w_j| \leq e^{\beta_j}$, the Laurent series is the corresponding power-series, with redundant coefficients being 0.

The function $f(z) = F(e^z)$ has the expansion

$$\Sigma_{n_j=-\infty}^{\infty} a_{n_1 \cdots n_k} e^{(n_1 z_1 + \cdots + n_k z_k)} \tag{12}$$

in the interior of T_R. If R_1 and R_2 are two fully intersecting rectangles, then by the uniqueness of Laurent expansions, the expansion which corresponds to any rectangle in the intersection $R_1 \cdot R_2$ must converge in T_{R_1} and T_{R_2}. Thus there exists one expansion for all of T. Conversely, the set over which a series (12) converges is a (general) tube, since it is the set over which

$$\Sigma_{n_j=-\infty}^{\infty} |a_{n_1 \cdots n_k}| e^{n_1 x_1 + \cdots + n_k x_k} \tag{13}$$

is finite, and thus the largest open set over which (12) converges is an open tube.

Theorem 8. *The point set over which a series* (13) *is convergent is a convex set.*

Every (open) convex tube is the domain of regularity of a periodic function with an expansion (12).

Proof. If $A \geq 0$, $B \geq 0$, and $0 \leq \vartheta \leq 1$, then $A^\vartheta B^{1-\vartheta} \leq (A + B)^\vartheta \cdot (A + B)^{1-\vartheta} = A + B$. Hence, if $x' = (x_1', \cdots, x_k')$ and $x'' = (x_1'', \cdots, x_k'')$ are any real points, and if x is the point with the components $x_j = \vartheta x_j' + (1 - \vartheta)x_j''$, then

$$\begin{aligned} &|a_{n_1 \cdots n_k}| e^{n_1 x_1 + \cdots + n_k x_k} \\ &\qquad = (|a_{n_1 \cdots n_k}| e^{n_1 x_1' + \cdots + n_k x_k'})^\vartheta \, (|a_{r_1 \cdots n_k}| e^{n_1 x_1'' + \cdots + n_k x_k''})^{1-\vartheta} \\ &\qquad\qquad \leq |a_{n_1 \cdots n_k}| e^{n_1 x_1' + \cdots + n_k x_k'} + |a_{n_1 \cdots n_k}| e^{n_1 x_1'' + \cdots + n_k x_k''} \end{aligned}$$

Thus, if (13) is convergent for two points it is also convergent for all points of the conjoining segment.

As for the second half of the theorem, by Theorem 1 it is sufficient to show that corresponding to any boundary point $\zeta_j = \xi_j + i\eta_j$ of the convex tube there exists a series (12) which is convergent in T but unbounded at ζ. It is easy to see that for that purpose we may translate each component by an imaginary distance, hence we may

assume that our point is a real point $(\xi_1, \cdots, \xi_k)$. Since $(\xi_1, \cdots, \xi_k)$ is a boundary point of the convex basis S there exists a supporting plane

$$L(x) \equiv \alpha_1 x_1 + \cdots + \alpha_k x_k + \gamma = 0$$

such that for $x \in S$, $L(x) < 0$, but $L(\xi) = 0$. Corresponding to $n = 0, 1, 2, \cdots$ we introduce integers $p_j(n)$ for which

$$|p_j(n) - n\alpha_j| \leq 1$$

Obviously

$$|nL(x) - [p_1(n)x_1 + \cdots + p_k(n)x_k + n\gamma]| \leq |x_1| + \cdots + |x_k|$$

and hence

$$\Sigma_{n=0}^{\infty} e^{n\gamma} e^{p_1(n)x_1 + \cdots + p_k(n)x_k} = \rho(x) \Sigma_{n=0}^{\infty} e^{nL(x)} \tag{14}$$

where

$$e^{-|x_1| - \cdots - |x_k|} \leq \rho(x) \leq e^{|x_1| + \cdots + |x_k|}$$

The multi-indices $(p_1(n), \cdots, p_k(n))$ occurring on the left side of (14) need not be different from each other, but collecting terms of the same multi-index we obtain

$$\Sigma_{n_j=-\infty}^{\infty} a_{n_1 \cdots n_k} e^{n_1 x_1 + \cdots + n_k x_k} \equiv \rho(x) \Sigma_{n=0}^{\infty} e^{nL(x)}$$

the coefficients of the left series being nonnegative. Since $L(x) < 0$ in S, but $L(x) \to 0$ as $(x) \to (\xi)$, we see that the corresponding series (12) with the nonnegative coefficients $\{a\}$ just obtained converges in T but its sum is unbounded at (ξ).

Theorem 9. *Every tube has a uniquely determined largest analytic completion. It is the convex hull of the given tube.*

Thus far we have proved this theorem for completions relative to multiperiodic functions. But we will prove in the following sections that the theorem also holds for completions relative to any functions without restriction. We will have only to show that every analytic function in a tube has a continuation into its convex hull. The second statement of Theorem 9 will then follow from the second statement of Theorem 8.

§5. Analytic Functions in Elliptical Polycylinders

Denoting by $C(r_\alpha)$ the polycylinder

$$|t_\alpha| < r_\alpha, \qquad \alpha = 1, \cdots, k, \qquad r_\alpha > 0 \tag{15}$$

a slight adaptation of the argument in section 4 will readily give the following theorem.

Theorem 10. *If $F(t_1, \cdots, t_k)$ is analytic in the union of two polycylinders $C(r'_\alpha)$, $C(r''_\alpha)$ it is continuable into the union of all polycylinders $C(r_\alpha)$ where*

$$\log r_\alpha = \vartheta \log r'_\alpha + (1 - \vartheta) \log r''_\alpha, \qquad 0 \leq \vartheta \leq 1 \tag{16}$$

the continuation being effected by the expansion

$$\Sigma^\infty_{n_\alpha=0} a_{n_1 \cdots n_k} t_1^{n_1} \cdots t_k^{n_k} \tag{17}$$

of $F(t)$ *in the neighborhood of* $(t) = (0)$.

We are now going to generalize this theorem from circles to ellipses If $t = re^{i\varphi}$, $r > 1$, and

$$z = \frac{1}{2}\left(t + \frac{1}{t}\right) \tag{18}$$

then

$$z = \frac{1}{2}\left(r + \frac{1}{r}\right) \cos \varphi + \frac{1}{2} i \left(r - \frac{1}{r}\right) \sin \varphi \tag{19}$$

Thus the circle $|t| = r$, $r > 1$, maps into the ellipse

$$\frac{x^2}{\frac{1}{4}\left(r + \frac{1}{r}\right)^2} + \frac{y^2}{\frac{1}{4}\left(r - \frac{1}{r}\right)^2} = 1 \tag{20}$$

with foci at $(\pm 1, 0)$ and r as the sum of the two semiaxes. Since

$$t = z + (z^2 - 1)^{\frac{1}{2}}$$

where, for large z,

$$(z^2 - 1)^{\frac{1}{2}} = z + 0\left(\frac{1}{z}\right)$$

the ellipse (20) is

$$|z + (z^2 - 1)^{\frac{1}{2}}| = r$$

We also introduce its interior

$$E(r): \qquad |z + (z^2 - 1)^{\frac{1}{2}}| < r \tag{21}$$

We now consider a second set of variables

$$\zeta = \frac{1}{2}\left(\tau + \frac{1}{\tau}\right), \qquad \tau = \zeta + (\zeta^2 - 1)^{\frac{1}{2}} \tag{22}$$

and we state the following lemma

Lemma 1. *If*

$$R_0(z) = 1, \qquad R_n(z) = t^n + t^{-n}, \qquad n \geq 1 \tag{23}$$

$$H_n(\zeta) = \frac{1}{2}\frac{\tau}{\tau^2 - 1} \cdot \frac{1}{\tau^n} \tag{24}$$

then

$$|R_n(z)| \leq 2r^n, \text{ for } z \text{ on the boundary of } E(r), \qquad r > 1 \tag{25}$$

$$|H_n(\zeta)| \leq \frac{1}{2}\frac{r}{r^2 - 1}\frac{1}{r^n}, \text{ for } \zeta \text{ on the boundary of } E(r), \qquad r > 1 \tag{26}$$

For $1 \leq |t| < |\tau|$ *we have*

$$\frac{1}{\zeta - z} = \Sigma_{n=0}^{\infty} R_n(z) H_n(\zeta) \tag{27}$$

the series being absolutely and uniformly convergent if, for positive numbers δ *and* M,

$$1 \leq |t| < |\tau| - \delta \leq M$$

Proof. Relations (25), (26) follow immediately from the definitions. The principal statement of the lemma follows from the (admissible) expansions

$$\frac{1}{\zeta - z} = \frac{1}{2}\frac{1}{\tau + \dfrac{1}{\tau} - t - \dfrac{1}{t}} = \frac{1}{2}\frac{1}{\tau - t - \dfrac{\tau - t}{\tau t}}$$

$$= \frac{1}{2}\frac{\tau t}{(\tau - t)\tau t - (\tau - t)} = -\frac{1}{2}\frac{t}{\left(t - \dfrac{1}{\tau}\right)(t - \tau)}$$

$$= \frac{1}{2}\frac{t}{\tau - \dfrac{1}{\tau}}\left[\frac{1}{t - \dfrac{1}{\tau}} - \frac{1}{t - \tau}\right]$$

$$= \frac{1}{2}\frac{\tau}{\tau^2 - 1}\left[\Sigma_{n=0}^{\infty}\frac{1}{t^n\tau^n} + \Sigma_{n=1}^{\infty}\frac{t^n}{\tau^n}\right]$$

$$= \frac{1}{2}\frac{\tau}{\tau^2 - 1}\left[1 + \Sigma_{n=1}^{\infty}\frac{t^n + t^{-n}}{\tau^n}\right]$$

The lemma leads to the following theorem:

Theorem 11. *Let* $f(z)$ *be analytic in an elliptic polycylinder*

$$E(r_\alpha): \quad |z_\alpha + (z_\alpha^2 - 1)^{\frac{1}{2}}| < r_\alpha, \qquad \alpha = 1, \cdots, k \tag{28}$$

where

(29) $$r_\alpha > 1, \qquad \alpha = 1, \cdots, k$$

Then $f(z)$ has a unique development of the form

(30) $$f(z_1, \cdots, z_k) = \Sigma_{n_\alpha=0}^{\infty} a_{n_1 \ldots n_k} R_{n_1}(z_1) \cdots R_{n_k}(z_k)$$

valid in $E(r_\alpha)$, the convergence being absolute and uniform in every interior polycylinder $E(\rho_\alpha)$, for which $1 < \rho_\alpha < r_\alpha$, $\alpha = 1, \cdots, k$, the coefficients satisfying

(31) $$|a_{n_1 \ldots n_k}| \leq K(\rho)\rho_1^{-n_1} \cdots \rho_k^{-n_k}$$

$(n_1, \cdots, n_k = 0, 1, 2, \cdots)$.

If $f(z)$ is analytic in the union of two polycylinders $E(r'_\alpha)$, $E(r''_\alpha)$, it is continuable into the union of all polycylinders $E(r_\alpha)$ where

(32) $$\log r_\alpha = \vartheta \log r'_\alpha + (1 - \vartheta) \log r''_\alpha, \qquad 0 \leq \vartheta \leq 1$$

the continuation being effected by the series (30).

Proof. By Cauchy's formula,

(33) $$f(z_1, \cdots, z_k) = \left(\frac{1}{2\pi i}\right)^k \int \cdots \int \frac{f(\zeta_1, \cdots, \zeta_k)d\zeta_1 \cdots d\zeta_k}{(\zeta_1 - z_1) \cdots (\zeta_k - z_k)}$$

where the integration extends over the boundaries of ellipses $E(\rho'_\alpha)$ with

(34) $$\rho_\alpha < \rho'_\alpha < r_\alpha$$

By Lemma 1 the series

$$\frac{1}{\zeta_\alpha - z_\alpha} = \Sigma_{n=0}^{\infty} R_n(z_\alpha)H_n(\zeta_\alpha)$$

converges absolutely and uniformly for (z_α) in $E(\rho_\alpha)$ and (ζ_α) on the boundary $B(\rho'_\alpha)$ of $E(\rho'_\alpha)$. We insert these series into (33) and integrate termwise, thus obtaining the series (30) with

$$a_{n_1 \ldots n_k} = \left(\frac{1}{2\pi i}\right)^k \int_{B(\rho_1')} \cdots \int_{B(\rho_k')} f(\zeta_1, \cdots, \zeta_k) \\ H_{n_1}(\zeta_1) \cdots H_{n_k}(\zeta_k)d\zeta_1 \cdots d\zeta_k$$

On using (26) we obtain the inequalities (31), with a constant K depending upon the ρ'_α rather than on the ρ_α themselves.

The uniqueness of the expansion follows now, on basis of (31),

from the uniqueness of the Laurent development

$$\Sigma_{n_1,\cdots,n_k=0}^{\infty} a_{n_1\cdots n_k}(t_1^{n_1} + t_1^{-n_1}) \cdots (t_k^{n_k} + t_k^{-n_k}), \qquad \frac{1}{r_\alpha} < |t_\alpha| < r_\alpha$$

Finally, if $f(z)$ is analytic in the elliptical polycylinders $E(r'_\alpha)$, $E(r''_\alpha)$, then, on the basis of (31), power-series (17), with the coefficients of the expansion (30), will likewise converge in the circular polycylinders $C(r'_\alpha)$ and $C(r''_\alpha)$. By Theorem 10, the power-series will converge in any $C(r_\alpha)$ satisfying (16). Hence, by (25), the expansion (30) will converge continuously in $E(r_\alpha)$, and this completes the proof of the theorem.

In the following section we shall replace the polycylinder $E(r_\alpha)$ by the polycylinder $E_l(r_\alpha)$ whose components are ellipses with foci at the points $z_\alpha = il$, $-il$, the quantity r_α being again the sum of the two semiaxes of the α-th ellipse. The quantity l is a fixed positive number. It is easily seen that the following statement remains valid.

Lemma 2. *If $f(z)$ is analytic in the union of $E_l(r'_\alpha)$ and $E_l(r''_\alpha)$ it is continuable in the union of the $E_l(r_\alpha)$ which belong to the family* (32).

§6. Completion of Tubes. A Special Case

Let us consider a tube whose basis S is the sum of two k-dimensional rectangles

$$|x_\alpha| < a'_\alpha, \qquad \alpha = 1, \cdots, k \tag{35}$$

$$|x_\alpha| < a''_\alpha, \qquad \alpha = 1, \cdots, k \tag{36}$$

Since a linear transformation of the x-coordinates can be interpreted as a linear transformation of the z-coordinates and carries tubes into tubes we are actually dealing with the case of a basis S which is the sum of two rectangles which are concentric and coaxial.

The tubes (35) and (36), that is the tubes with these bases, contain respectively the elliptic polycyclinders

$$E_l(r'_\alpha): \qquad \frac{x_\alpha^2}{a'^2_\alpha} + \frac{y_\alpha^2}{a'^2_\alpha + l^2} < 1, \qquad \alpha = 1, \cdots, k$$

$$E_l(r''_\alpha): \qquad \frac{x_\alpha^2}{a''^2_\alpha} + \frac{y_\alpha^2}{a''^2_\alpha + l^2} < 1, \qquad \alpha = 1, \cdots, k$$

with

$$r'_\alpha = a'_\alpha + \sqrt{a'^2_\alpha + l^2}, \qquad r''_\alpha = a''_\alpha + \sqrt{a''^2_\alpha + l^2}$$

this being true for every positive value of l. Thus by Lemma 2,

$f(z)$ is analytic also in the polycylinder $E_l(r_\alpha)$ whose minor semiaxes a_α satisfy the relations

$$\log [a_\alpha + \sqrt{a_\alpha^2 + l^2}] \\ = \vartheta \log [a'_\alpha + \sqrt{a'^2_\alpha + l^2}] + (1 - \vartheta) \log [a''_\alpha + \sqrt{a''^2_\alpha + l^2}]$$

or, what is the same,

$$\log \left[\frac{a_\alpha}{l} + \sqrt{\frac{a_\alpha^2}{l^2} + 1}\right] \\ = \vartheta \log \left[\frac{a'_\alpha}{l} + \sqrt{\frac{a'^2_\alpha}{l^2} + 1}\right] + (1 - \vartheta) \log \left[\frac{a''_\alpha}{l} + \sqrt{\frac{a''^2_\alpha}{l^2} + 1}\right] \tag{37}$$

If we let $l \to \infty$, relation (37) goes over into

$$a_\alpha = \vartheta a'_\alpha + (1 - \vartheta) a''_\alpha, \qquad 0 \leq \vartheta \leq 1 \tag{38}$$

and the polycylinder $E_l(r_\alpha)$ converges to the corresponding tube

$$|x_\alpha| < a_\alpha, \qquad \alpha = 1, \cdots, k$$

The totality of these tubes for all ϑ in $0 \leq \vartheta \leq 1$ is the convex hull of the tubes (35) and (36) and we thus obtain the following special case of Theorem 9.

Lemma 3. *If $f(z)$ is analytic in a tube T whose basis S consists of the sum of two rectangles which are concentric and coaxial, then $f(z)$ is analytic in the convex hull of T.*

§7. The General Case

The transition from the special base S as in Lemma 3 to the most general domain S will be purely point-theoretical. We will carry it out in three steps.

Step 1°. *Crescent-shaped wedges.* Consider first, for $k = 2$, a domain S in the (x, y)-plane which is bounded by two *convex* curves $y = \varphi(x)$, $y = \psi(x)$, with $\varphi(0) = \varphi(1) = \psi(0) = \psi(1) = a$, and $\varphi(x) < \psi(x)$ for $0 < x < 1$. (See Figure 1.) We will vary $\psi(x)$, and denote S explicitly by S_ψ. If $\psi_1 \geq \psi$ and $\psi_2 \geq \psi$, ($\psi_1(x)$ and $\psi_2(x)$ being convex functions) and if an analytic function in the tube $T(S_\psi)$ can be continued into both $T(S_{\psi_1})$ and $T(S_{\psi_2})$, then, as the reader will verify without difficulty, these continuations merge into one (one-valued) continuation into $T(S_{\psi_3})$ where $\psi_3 = \max(\psi_1, \psi_2)$. Also $\psi_3(x)$ is again convex. Now consider all convex functions $\bar{\psi}(x) \geq \psi(x)$ such that $T(S_{\bar{\psi}})$ is an analytic completion of $T(S_\psi)$ and form the

function, again convex,

$$\psi^*(x) = \sup_{\tilde{\psi}} \tilde{\psi}(x) \tag{39}$$

Obviously $T(S_{\psi^*})$ is an analytic completion of $T(S_\psi)$ and we claim that S_{ψ^*} is the convex hull of S_ψ, that is,

$$\psi^*(x) \equiv a \tag{40}$$

If this were not the case we could construct a configuration, as in Figure 2. The point C is the minimum nearest to $(0, a)$ of $\psi^*(x)$,

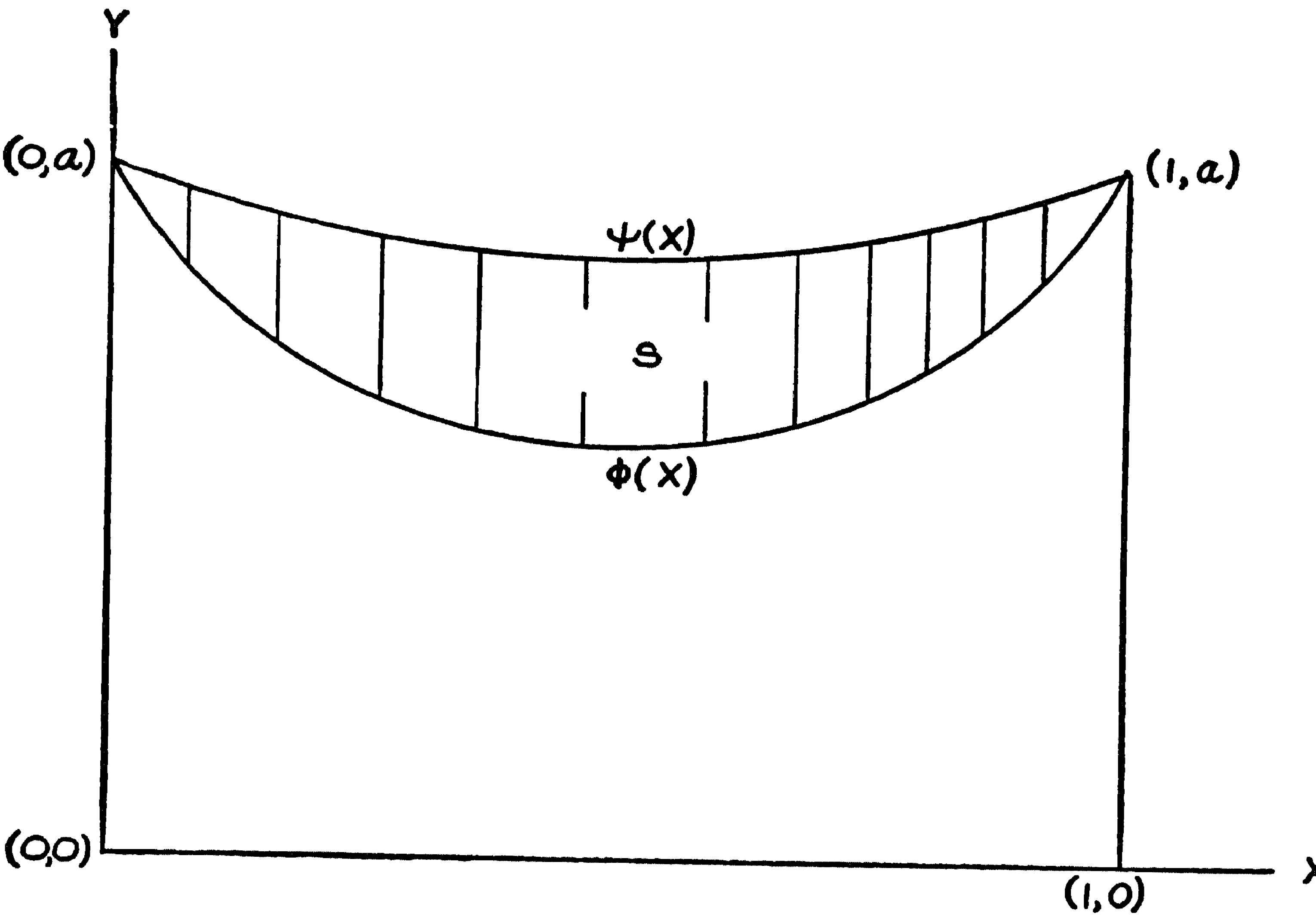

Figure 1.

and D is a point sufficiently near to C so as to enable the insertion in S_{ψ^*} of two coaxial and concentric rectangles R as in the figure. By Lemma 3, the tube with basis the convex hull $\tilde{R}$ of R is an analytic completion of $T(R)$, and since $\tilde{R}$ intersects S_{ψ^*} in a domain it follows that the tube with basis $\tilde{R} + S_{\psi^*}$ is an analytic completion of $T(S_{\psi^*})$ and hence of $T(S_\psi)$. However, the domain $\tilde{R} + S_{\psi^*}$ is bounded by a convex function $\tilde{\psi}(x) \geq \psi^*(x)$, and is genuinely greater than S_{ψ^*} since the point C is not contained in S_{ψ^*} but is in $S_{\tilde{\psi}}$, which is contradictory to the definition of $\psi^*(x)$. This yields (40) for the case $k = 2$.

For $k \geq 3$ we call the variables $x, y, x_3, \cdots, x_k$ and define a crescent-shaped wedge as a 2-dimensional crescent in (x, y)-coordinates as described before multiplied topologically with a $(k-2)$-dimensional cube

$$|x_3| < \delta, \qquad \cdots, \qquad |x_k| < \delta \tag{41}$$

The previous argument extends to cover this case as well. Thus the

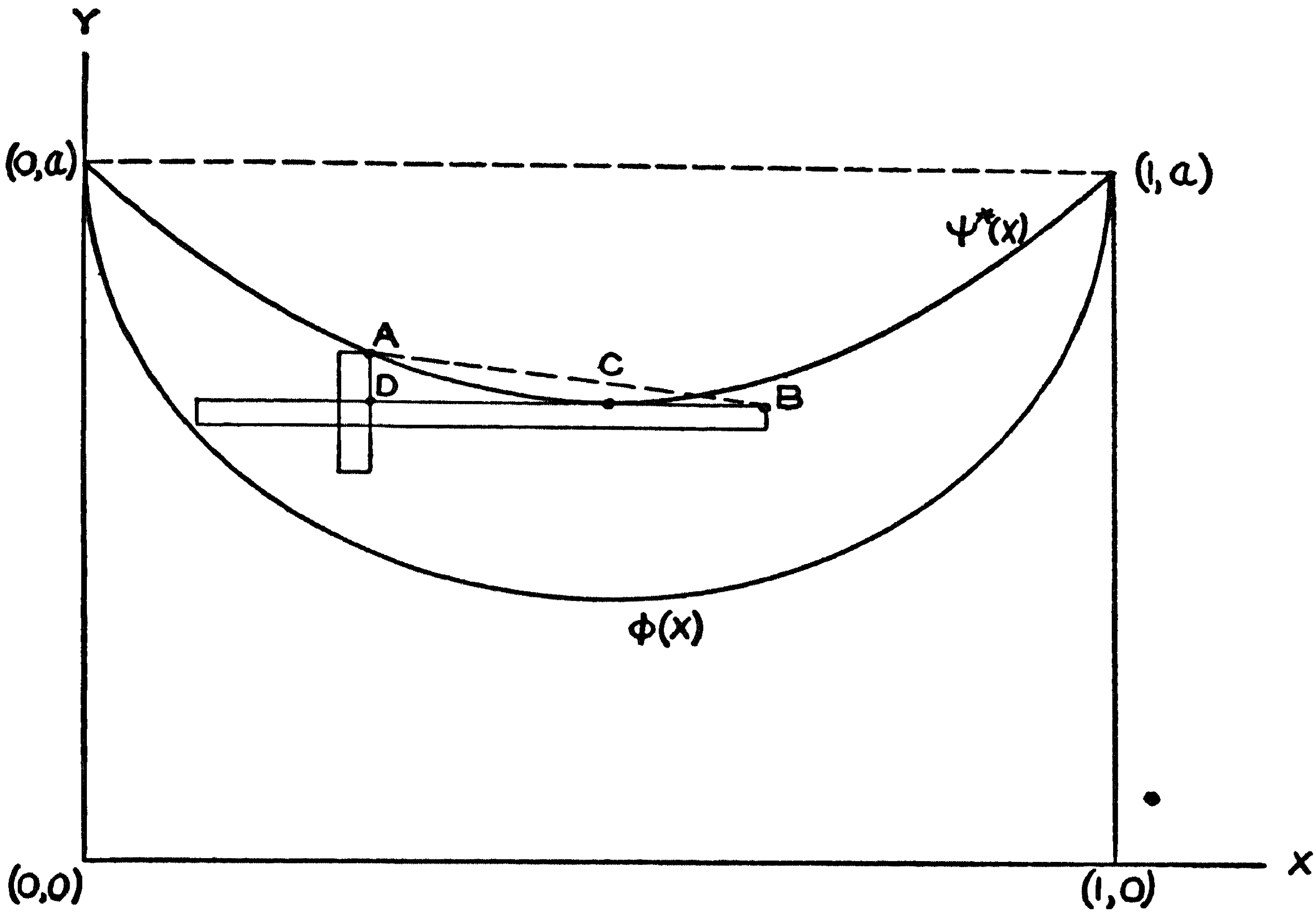

Figure 2.

analytic completion of the tube with basis

$$[(x, y) \in S_\psi, \qquad |x_3| < \delta, \qquad \cdots, \qquad |x_k| < \delta]$$

is the tube with basis

$$[(x, y) \in S_{\psi^*}, \qquad |x_3| < \delta, \qquad \cdots, \qquad |x_k| < \delta]$$

where $\psi^*(x) \equiv a$ as in (40).

Step 2°. *Union of two intersecting convex domains.* Assume that S is the union of two intersecting convex domains S_1 and S_2, all in $(x_1, \cdots, x_k)$-space. See Figure 3. Take any point P_1 in S_1, a point P_2 in S_2, and a point O in the intersection $S_1 \cap S_2$. In the

triangle P_1OP_2 take a point O' sufficiently near O and in a suitably chosen oblique coordinate system identify the curves P_1OP_2 and $P_1O'P_2$ with the curves $\varphi(x)$ and $\psi(x)$ respectively. For $k \geq 3$ multiply topologically by a small cube of $(k-2)$-dimensions. If $f(z)$ is analytic in $T(S)$ then it can be continued analytically from the tube with basis the wedge $P_1OP_2O'P_1$ into the tube with basis the "triangle" P_1OP_2 (with $(k-2)$-dimensional thickness) and since this

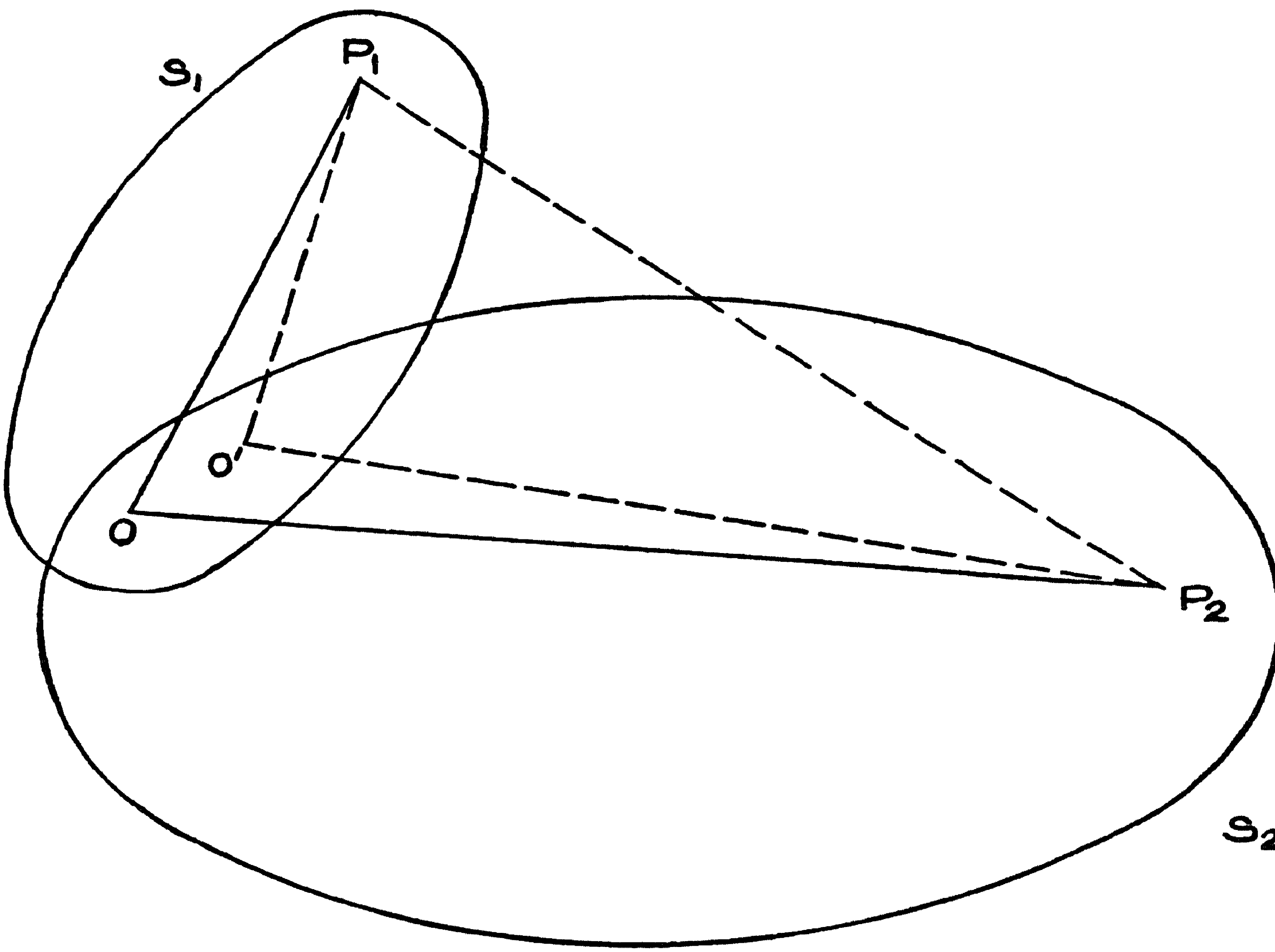

Figure 3.

triangle intersects S in a domain, this is a genuine continuation. Any two "triangles," being convex, intersect in domains, or not at all, thus by Theorem 5, Chapter II, we may continue $f(z)$ from the tube with basis S into the tube with basis the union of all triangles P_1OP_2, with P_1 contained in S_1, P_2 in S_2 and O in $S_1 \cap S_2$. We now claim that the union of S with all such triangles is the convex hull $\bar{S}$ of S. It is obviously part of the convex hull. Thus it will be sufficient to show that the union S^* of all possible segments P_1P_2 constitutes a convex point set. For this purpose let us consider any two points P

and Q in S^*. We shall show that every point R on the segment PQ lies in S^*. By construction there exist two points P_1, Q_1 in S_1 and two points P_2, Q_2 in S_2 such that P lies on the segment P_1P_2 and Q

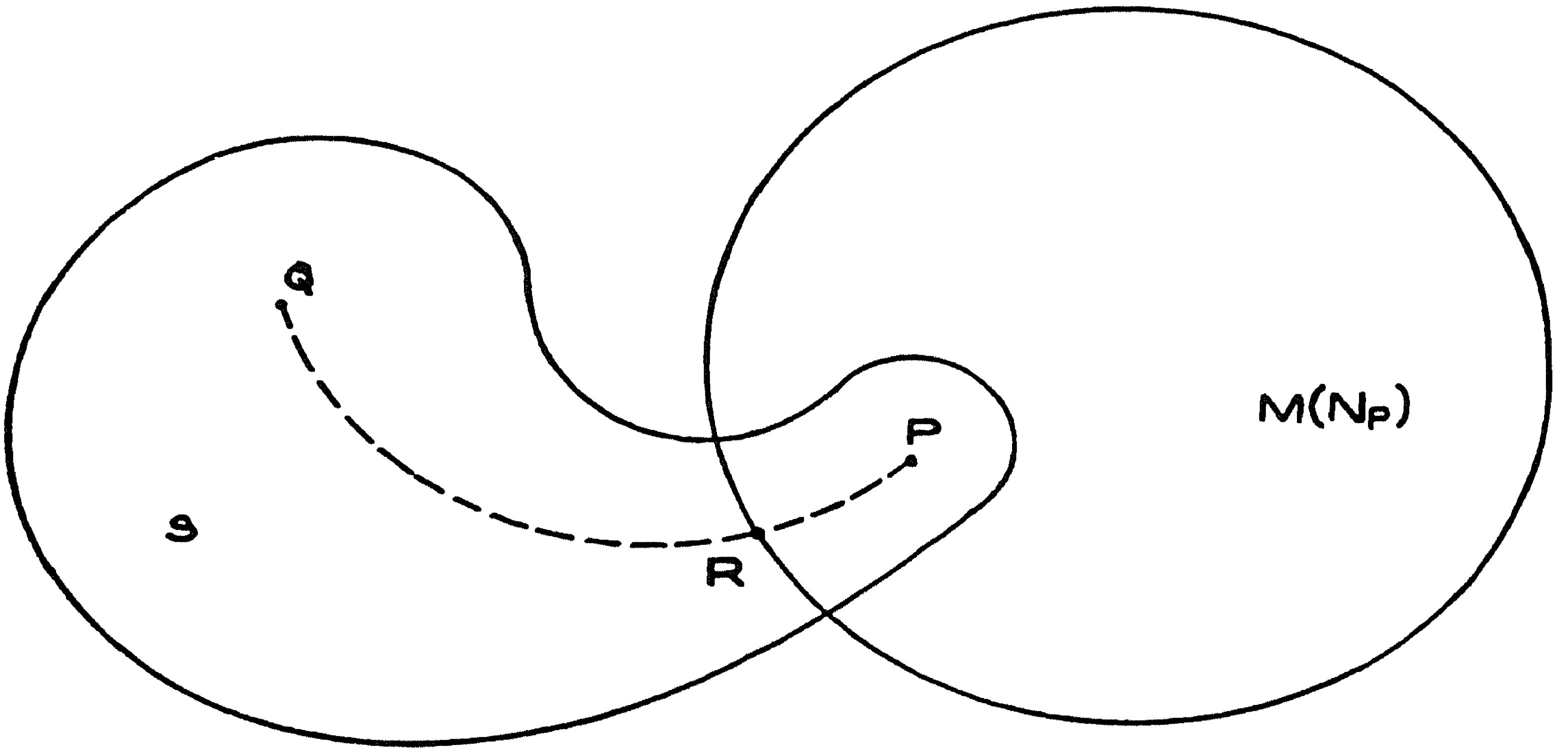

Figure 4.

lies on the segment Q_1Q_2. In vector notation this means

$$\begin{aligned} P &= \lambda_1 P_1 + \lambda_2 P_2 \qquad (0 \leq \lambda \leq 1,\ \lambda_2 = 1 - \lambda_1) \\ Q &= \mu_1 Q_1 + \mu_2 Q_2 \qquad (0 \leq \mu_1 \leq 1,\ \mu_2 = 1 - \mu_1) \end{aligned}$$

Now any point R on the segment PQ is given by

$$\begin{aligned} R &= \rho P + \sigma Q \qquad (0 \leq \rho \leq 1,\ \sigma = 1 - \rho) \\ &= (\rho\lambda_1 P_1 + \sigma\mu_1 Q_1) + (\rho\lambda_2 P_2 + \sigma\mu_2 Q_2) \\ &= (\rho\lambda_1 + \sigma\mu_1)\left[\frac{\rho\lambda_1}{\rho\lambda_1 + \sigma\mu_1} P_1 + \frac{\sigma\mu_1}{\rho\lambda_1 + \sigma\mu_1} Q_1\right] \\ &\qquad + (\rho\lambda_2 + \sigma\mu_2)\left[\frac{\rho\lambda_2}{\rho\lambda_2 + \sigma\mu_2} P_2 + \frac{\sigma\mu_2}{\rho\lambda_2 + \sigma\mu_2} Q_2\right] \\ &= (\rho\lambda_1 + \sigma\mu_1) R_1 + (\rho\lambda_2 + \sigma\mu_2) R_2 \end{aligned}$$

where R_1 is a point on the segment P_1Q_1 (and hence in S_1) and R_2 is a point on P_2Q_2 (and hence in S_2). But since

$$(\rho\lambda_1 + \sigma\mu_1) + (\rho\lambda_2 + \sigma\mu_2) = \rho(\lambda_1 + \lambda_2) + \sigma(\mu_1 + \mu_2) = \rho + \sigma = 1$$

it follows that R lies on the line segment R_1R_2 and hence by construction must lie in S^*. Thus S^* is convex by the definition of convexity.

Step 3°. *Arbitrary domains.* Let S be any domain in $(x_1, \cdots, x_k)$-space, let P be any point in S, and $N = N(P)$ a neighborhood of P contained in S. Let $M = M(N)$ be a maximal convex domain containing N and such that the tube $T(M)$ is an analytic completion of the tube $T(N)$. We shall show that M contains the convex hull $\tilde{S}$ of S,

$$\tilde{S} \subset M \tag{42}$$

If (42) does not hold, then there must be a point Q in S not in M. (See Figure 4.) Join Q to P by an arc lying in S, and let R be a point in which this arc cuts the boundary of M. Let $N' = N'(R)$ be a neighborhood of R contained in S and let $M' = M'(N')$ be a maximal convex basis containing N'. Then $T(M \cup M')$ is an analytic completion of $T(S)$ and the two convex domains M and M' clearly intersect (near R). Thus by Step 2° the tube T with basis their convex hull is an analytic completion of $T(N)$, but this contradicts the maximality property of M. Thus (42) holds and Theorem 9 is proved in its full generality.

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CHAPTER VI

Inequalities, Bounds, and Norms

§1. INEQUALITY OF JENSEN-HARTOGS

Consider a function $f(z)$ of one complex variable $z = x + iy$. If the function is analytic in (a neighborhood of)

(1) $$|z| \leq \rho$$

and if we introduce the Poisson kernel

(2) $$P(z, \zeta) = \frac{1}{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\varphi - \alpha) + r^2}$$

where

(3) $$z = re^{i\varphi}, \qquad \zeta = \rho e^{i\alpha}, \qquad 0 \leq r < \rho$$

then the inequality we have in mind is

(4) $$\log |f(z)| \leq \int_0^{2\pi} P(z, \zeta) \log |f(\zeta)| d\alpha$$

For the proof we first note that the inequality is additive: if it holds for $f_1(z)$ and $f_2(z)$ it also holds for $f_1 f_2$ since $\log |f_1 f_2| = \log |f_1| + \log |f_2|$. Next, the function $f(z)$ has at most a finite number of zeros inside and on (1). Denoting the zeros in $|z| < \rho$ by $a_1, \cdots, a_p$, and those on $|z| = \rho$ by $b_1, \cdots, b_q$, with multiple zeros occurring a number of times corresponding to their multiplicities, we write $f(z)$ as the product

(5) $$f(z) \equiv g(z) \Pi_{m=1}^{p} \frac{z - a_m}{\rho^2 - \bar{a}_m z} \Pi_{n=1}^{q} (z - b_n)$$

In this factorization, $g(z)$ is defined by the relation stated, as the quotient of $f(z)$ with the other factors. We will now verify (4) for each factor separately. The factor $g(z)$ is analytic and has no zeros in (1), and so $\log |g(z)| \equiv \mathrm{Re}[\log g(z)]$ is a harmonic function in (1); therefore, (4) holds, as a strict equality, on the basis of Poisson's formula for harmonic functions. In the case of a factor

$$\varphi(z) = \frac{z - a}{\rho^2 - \bar{a}z}, \qquad |a| < \rho$$

we have $|\varphi(z)| < 1$, that is $\log |\varphi(z)| < 0$, in $|z| < \rho$, and $|\varphi(\zeta)| = 1$, that is $\log |\varphi(\zeta)| = 0$, for $|\zeta| = \rho$; therefore (4) holds as a genuine

inequality. Finally in the case of a factor $\psi(z) = z - b$, $b = \rho e^{i\beta}$, we may first assume, replacing z by zb, that $\rho = 1$ and $b = 1$, and thus it will suffice to prove that

$$\log |z - 1| = \int_0^{2\pi} P(z, e^{i\alpha}) \log |e^{i\alpha} - 1| d\alpha \tag{6}$$

for $|z| < 1$.

For any $\epsilon > 0$, the function $z - 1 - \epsilon$ has no zeros in $|z| \leq 1$, and thus we may start from the relation

$$\log |z - 1 - \epsilon| = \int_0^{2\pi} P(z, e^{i\alpha}) \log |e^{i\alpha} - 1 - \epsilon| d\alpha \tag{7}$$

Letting $\epsilon \to 0$, in order to derive (6) from (7) we will apply the following property of Lebesgue integral (the principle of dominated convergence). If $g_\epsilon(\alpha) \to g(\alpha)$, for almost all α in $a \leq \alpha \leq b$, and if $|g_\epsilon(\alpha)| \leq h(\alpha)$ where $\int_a^b h(\alpha) d\alpha < \infty$, then

$$\int_a^b g_\epsilon(\alpha) d\alpha \to \int_a^b g(\alpha) d\alpha$$

In our case, since

$$0 \leq P(z, \zeta) \leq \frac{1}{2\pi} \frac{\rho + r}{\rho - r} \tag{8}$$

it will be enough to verify that, for $0 < \epsilon \leq 1$,

$$\big|\log |e^{i\alpha} - 1 - \epsilon|\big| \leq h(\alpha) \tag{9}$$

where

$$\int_0^{2\pi} h(\alpha) d\alpha < \infty \tag{10}$$

Since

$$2\left|\sin \frac{\alpha}{2}\right| \leq |e^{i\alpha} - 1 - \epsilon| \leq 2 + \epsilon$$

relation (9) will be satisfied for

$$h(\alpha) = \left|\log 2\left|\sin \frac{\alpha}{2}\right|\right| + \log 3$$

and this obviously satisfies (10). This completes the proof of the inequality (4).

Our next aim is to generalize formula (4) to functions of several variables. In this connection it is appropriate to consider an enlarged concept of Lebesgue integral with free use of the real value $-\infty$. If $g(\alpha)$ is defined on an interval $a \leq \alpha \leq b$, and if its values are contained in a half line $-\infty \leq g(\alpha) \leq C$, for some finite C, then we call

$g(\alpha)$ measurable, if for every λ, the set of points for which $g(\alpha) \leq \lambda$ is a measurable set. In particular if $g(\alpha)$ is measurable, then the point set on which $g(\alpha) = -\infty$ is measurable, and its measure may be > 0. With $g(\alpha)$ we can associate an integral $\int_a^b g(\alpha)d\alpha$ which is a finite number or $-\infty$. It is additive, and linear with respect to positive coefficients. Furthermore it has the following important property. If

$$g_n(\alpha) \downarrow g(\alpha) \tag{11}$$

then

$$\int_a^b g_n(\alpha)d\alpha \downarrow \int_a^b g(\alpha)d\alpha \tag{12}$$

where "$\downarrow$" means converges decreasingly to. In particular, if $g(\alpha)$ is given, we can find a sequence of functions $g_n(\alpha)$, each bounded, such that (11) holds and thus our enlarged Lebesgue integral is, for each function, a limit of integrals of bounded functions. The first use we make of the enlarged concept is to admit in relation (4), and in the subsequent generalization to several variables, the function $f(z) \equiv 0$. For this function both sides in (4) have value $-\infty$, and the relation holds.

Our enlarged concept applies to any Lebesgue integral, and in particular also to functions in several variables. Now let $g(\alpha, \beta)$ be defined and measurable in a square $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, and of course $-\infty \leq g(\alpha, \beta) \leq C$, for some C. The important fact is that Fubini's theorem again holds. It states that the repeated integral

$$\int_0^1 d\alpha \int_0^1 g(\alpha, \beta)d\beta \tag{13}$$

likewise exists, and that it has the same value as the multiple integral

$$\int_0^1 \int_0^1 g(\alpha, \beta)d\alpha d\beta \tag{14}$$

whether finite or $-\infty$. Furthermore the theorem also holds if α, β are representative of collections of Euclidean variables, say.

We now consider the case of k complex variables. Putting

$$z_j = r_j e^{i\varphi_j}, \qquad \zeta_j = \rho_j e^{i\alpha_j}, \qquad j = 1, \cdots, k \tag{15}$$

$$P_k(z, \zeta) \equiv \Pi_{j=1}^k P(z_j, \zeta_j), \qquad 0 \leq r_j < \rho_j \tag{16}$$

we claim that for any function $f(z_1, \cdots, z_k)$ which is analytic in

$$|z_j| \leq \rho_j, \qquad j = 1, \cdots, k \tag{17}$$

the relation

$$\log |f(z)| \leq \int_0^{2\pi} \cdots \int_0^{2\pi} P_k(z, \zeta) \log |f(\zeta)| d\alpha_1 \cdots d\alpha_k \tag{18}$$

holds. First of all $\log |f(\rho_1 e^{i\alpha_1}, \cdots, \rho_k e^{i\alpha_k})|$ is a descending limit as $\epsilon \downarrow O$ of the continuous functions

$$g_\epsilon(\alpha) = \log (|f(\rho_1 e^{i\alpha_1}, \cdots, \rho_k e^{i\alpha_k})| + \epsilon)$$

and thus it is measurable in $\alpha_1, \cdots, \alpha_k$. Now, for $k = 1$, (18) has been proved. If it is known to hold for some k, and if $f(z; z_{k+1}) \equiv f(z_1, \cdots, z_k; z_{k+1})$ is a function of $k + 1$ variables, we have first of all, by (18)

$$\text{(19)} \quad \log |f(z; z_{k+1})| \leq \int_0^{2\pi} \cdots \int_0^{2\pi} P_k(z; \zeta) \log |f(\zeta; z_{k+1})| d\alpha_1 \cdots d_{\alpha_k}$$

By (4) we have

$$\log |f(\zeta; z_{k+1})| \leq \int_0^{2\pi} P(z_{k+1}, \zeta_{k+1}) \log |f(\zeta; \zeta_{k+1})| d\alpha_{k+1}$$

and on substituting this in (19), and applying Fubini's theorem, we obtain relation (18) for $k + 1$.

In certain cases it is useful to have a counterpart to the estimate (8), namely

$$\text{(20)} \qquad |P_k(z, \zeta)| \geq \left(\frac{1}{2\pi}\right)^k \Pi_{j=1}^k \frac{\rho_j - r_j}{\rho_j + r_j} \equiv M_k(\rho; r)$$

If we assume that $|f(\zeta)| \leq 1$, that is

$$\text{(21)} \qquad \log |f(\zeta)| \leq 0$$

then we can deduce from (18) that

$$\text{(22)} \qquad \log |f(z)| \leq M_k(\rho; r) \int_0^{2\pi} \cdots \int_0^{2\pi} \log |f(\zeta)| dv_\alpha$$

where $dv_\alpha \equiv d\alpha_1 \cdots d\alpha_k$. If A is any measurable set on the k-dimensional torus

$$0 \leq \alpha_j < 2\pi, \qquad j = 1, \cdots, k$$

then, under the assumption (21) we certainly can replace (22) by the weaker inequality

$$\log |f(z)| \leq M_k(\rho; r) \int_A \log |f(\zeta)| dv_\alpha$$

Now, if $|f(\rho_j e^{i\alpha_j})| \leq \epsilon < 1$ on A, and mA denotes the measure of the set A, we hence obtain

$$\log |f(z)| \leq -M_k(\rho; r) \cdot mA \cdot \log \frac{1}{\epsilon}$$

Hence we may conclude: if the functions $f_n(z)$, $n = 1, 2, \cdots$, are analytic and collectively bounded in (17), and if they are uniformly

convergent to 0 on a measurable subset A of the torus

$$|\zeta_j| = \rho_j, \qquad 0 \leq \alpha_j < 2\pi \tag{23}$$

and if A has positive measure, then the sequence $f_n(z)$ is continuously convergent to 0 in the interior of (17).

By a conformal mapping of each component D_j onto a circle the reader will hence derive the following theorem:

Theorem 1. *If $D = D_1 \times \cdots \times D_k$ is a polycylinder, if the boundary of D_j is a Jordan curve C_j, and if C'_j is an open arc of C_j; if the functions $f_n(z)$ are analytic and collectively bounded in the closure of D, and uniformly convergent to 0 on the subset $C'_1 \times \cdots \times C'_k$ of the boundary of D, then the functions are continuously convergent to 0 in D.*

We will now formulate a very specialized conclusion from this theorem for later use. If H is a convex domain in real $(x_1, \cdots, x_k)$-space, and $(0, \cdots, 0)$ is a point of its boundary, if T_H is the tube with basis H, and N is a (convex) domain in (z)-space containing the origin, if $g(z)$ is analytic in N and if the functions $f_n(z)$ are analytic and boundedly convergent in the closure of T_H and uniformly convergent to $g(\zeta)$ on the intersection of N with the k-dimensional manifold

$$\operatorname{Re} \zeta_1 = 0, \cdots, \operatorname{Re} \zeta_k = 0 \tag{24}$$

(which lies on the boundary of T_H), then the limit $f(z)$ of the sequence $f_n(z)$ is an analytic continuation of $g(z)$ from N onto $N + T_H$. In fact, since H is convex, and (0) a boundary point, there exists a parallelepiped A in x-space with (0) as one of its vertices, whose closure but for that vertex is contained in H. A real homogenous transformation will make A into a parallelogram

$$A: \qquad 0 < x_j < \alpha_j, \qquad j = 1, \cdots, k$$

We now construct a $2k$-dimensional parallelogram

$$(x) \in A, \qquad -a < y_j < a, \qquad j = 1, \cdots, k$$

For A and a sufficiently small, it is contained in N. It also is of the form $D_1 \times \cdots \times D_k$ as in Theorem 1, and a portion of (24) is of the form $C'_1 \times \cdots \times C'_k$. Our conclusion now follows by applying Theorem 1 to the functions

$$f_n(z) - g(z), \qquad n = 1, 2, \cdots$$

§2. Maximum on the Boundary

If $f(z)$ is analytic in a bounded domain D in $(z_1, \cdots, z_k)$ space and has continuous boundary values $f(\zeta)$ on its boundary B, then we

shall show that

$$|f(z)| \leq \sup_{\zeta \in B} |f(\zeta)|, \qquad z \in D \tag{25}$$

with equality holding at any one point of D only if

$$|f(z)| \equiv \text{const.} \tag{26}$$

In fact, if λ_0 is the maximum of $|f(z)|$ in $D + B$, and L is the point set on which $|f(z)| = \lambda_0$, then either $L \supset D$, and this implies (26); or L is contained entirely in B, thus implying (25). To see this we note that if the intersection $L \cap D$ were not vacuous, then the open domain $D - (L \cap D)$ would have a boundary point z^0 in D. This point would belong to L, and thus at this point $|f(z)|$ would have a weak relative maximum without being constant in the neighborhood of z^0. However this can be shown to be contradictory as follows. If we shift z^0 into the origin and apply Cauchy's formula (9), Chapter II, for the curves C_j: $|\zeta_j| = \rho_j$ with $z = 0$ and ρ_j sufficiently small, we obtain

$$f(0) = \left(\frac{1}{2\pi}\right)^k \int_0^{2\pi} \cdots \int_0^{2\pi} f(\rho_j e^{i\alpha_j}) dv_\alpha \tag{27}$$

hence

$$|f(0)| \leq \left(\frac{1}{2\pi}\right)^k \int_0^{2\pi} \cdots \int_0^{2\pi} |f(\rho_j e^{i\alpha_j})| dv_\alpha \tag{28}$$

and thus $|f(z)|$ cannot have a maximum at an interior point, without being constant in its neighborhood.

The same argument also yields (25) for an unbounded domain D, provided

$$|f(z)| \to 0 \qquad \text{as} \qquad |z_1|^2 + \cdots + |z_k|^2 \to \infty, \qquad \text{with} \qquad z \in D \tag{29}$$

Now, if D is a tube T_A with bounded basis A, and $f(z)$ is bounded with continuous boundary values $f(\zeta)$, then following Phragmen-Lindelof, we consider the functions

$$f_\epsilon(z) = f(z) e^{\epsilon(z_1^2 + \cdots + z_k^2)}$$

for $\epsilon > 0$. For each $f_\epsilon(z)$ we have (29) and hence (25). Now for $\epsilon \to 0$; $f_\epsilon(z) \to f(z)$ and

$$\lim_{\epsilon \to 0} \sup_{\zeta \in B} |f_\epsilon(\zeta)| \leq \sup_{\zeta \in B} |f(\zeta)|$$

and thus (25) holds for $f(z)$ itself.

Furthermore, for any real numbers $\alpha_1, \cdots, \alpha_k$, the function

$$f(z)e^{\alpha_1 z_1 + \cdots + \alpha_k z_k} \tag{30}$$

is again bounded with continuous boundary values in T_A, and thus on introducing the function

$$M_f(x) = \sup_{-\infty < y_j < \infty} |f(x_1 + iy_1, \cdots, x_k + iy_k)|$$

we see that for any real numbers $\alpha_1, \cdots, \alpha_k$ the function

$$\log M_f(x) + \alpha_1 x_1 + \cdots + \alpha_k x_k$$

assumes its supremum on the boundary of A, and thus is convex on any segment. From this we deduce

Theorem 2. *If Δ is a domain in $(x_1, \cdots, x_k)$-space, if A is any set in Δ, and if H is the convex hull of A, if H is likewise contained in Δ, and if $f(z)$ if analytic and bounded in T_Δ, then*

$$\sup_{x \in H} M_f(x) = \sup_{x \in A} M_f(x) \tag{31}$$

It is interesting to compare this result with what we already know from Theorem 9 of Chapter V. By that theorem if A is a *domain* in $(x_1, \cdots, x_k)$-space and if $f(z)$ is merely known to be analytic in the tube T_A, then f automatically has an analytic continuation into the convex hull T_H of T_A, and since f assumes in T_H only values which it assumes in T_A (see section 1, Chapter IV) it follows that (31) holds whenever f happens to be bounded in T_A.

In order to draw a pertinent conclusion from this theorem we will require some general facts on approximation of real function. We will assemble all the facts we will need in this and later chapters in the section to follow.

Before doing this we will just make a remark. It is easy to see from Cauchy's formula that for $f(z)$ analytic in (17), the maximum of $|f(z)|$ is assumed on the subset (23) of the entire boundary. Such subsets of the boundary of a domain have been first investigated by H. Poincaré, and more recently by S. Bergman (he calls them characteristic manifolds) and A. Weil. Obviously, such manifolds are manifolds of uniqueness. If a function vanishes on such a manifold, it must be identically zero. This suggests the possibility of constructing an analytic function from its values on the characteristic manifold, and this is the main problem in this field.

§3. APPROXIMATIONS

Let $K(t_1, \cdots, t_n)$ be a kernel possessing the following two properties:

1°. $\quad K(t_1, \cdots, t_n) \geq 0, \qquad -\infty < t_j < \infty$

2°. $\quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} K(t_1, \cdots, t_n)dt_1 \cdots dt_n = 1$

An example of such a kernel is

$$(32) \qquad K(t_1, \cdots, t_n) = \pi^{-\frac{n}{2}} e^{-(t_1^2 + \cdots + t_n^2)}$$

Let $f(x_1, \cdots, x_n)$ be bounded and measurable in $-\infty < x_j < \infty$. Then the functions

$$(33) \qquad f_\mu(x_1, \cdots, x_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\left(x_1 + \frac{t_1}{n}, \cdots, x_n + \frac{t_n}{n}\right) K(t)dv_t,$$

$$\mu = 1, 2, \cdots, \qquad dv_t \equiv dt_1 \cdots dt_n$$

certainly exist. Since they represent certain averages of f they may be expected to reflect various properties of f, and under certain circumstances they may converge to f. We shall give three lemmas relating to these matters.

Lemma 1. *Let $K(t)$ satisfy 1° and 2° and let $f(x_1, \cdots, x_n)$ be a bounded measurable function in $-\infty < x_j < \infty$. If in addition f is continuous in some domain D then*

$$(34) \qquad f_\mu \rightrightarrows f$$

in every closed set S contained in D, where "$\rightrightarrows$" means converges uniformly to.

Before giving the proof we make two very simple observations, based upon the property 2°. First

$$(35) \qquad f(x_1, \cdots, x_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, x_n)K(t_1, \cdots, t_n)dv_t$$

for every function f, and secondly

$$(36) \qquad \int \cdots \int_{\text{ext } C_N} K(t_1, \cdots, t_n)dv_t \to 0 \quad \text{as} \quad N \to \infty$$

where ext C_N denotes the exterior of the cube C_N,

$$(37) \qquad C_N: \qquad -N \leq t_j \leq N; \qquad j = 1, \cdots, n$$

We now pass to the proof of the lemma. On using (33) and (35) we have for x in S and $0 < \delta$

$$(38) \qquad |f_\mu(x) - f(x)| \leq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| f\left(x_1 + \frac{t_1}{\mu}, \cdots, x_n + \frac{t_n}{\mu}\right) - f(x_1, \cdots, x_n)\right| K(t)dv_t$$

$$\leq \int_{-\mu\delta}^{\mu\delta} \cdots \int_{-\mu\delta}^{\mu\delta} \left| f\left(x_j + \frac{t_j}{\mu}\right) - f(x_j) \right| K(t) dv_t$$
$$+ \int \cdots \int_{\text{ext } C_{\mu\delta}} \left| f\left(x_j + \frac{t_j}{\mu}\right) - f(x_j) \right| K(t) dv_t$$
$$\leq \max_{|t_j| \leq \delta} |f(x_1 + t_1, \cdots, x_n + t_n)$$
$$- f(x_1, \cdots, x_n)|$$
$$+ 2 \max_{-\infty < x_j < \infty} |f(x)| \cdot \int \cdots \int_{\text{ext } C_{\mu\delta}} K(t) dv_t$$

where in passing from the next to last to the last inequality we have used 2^0. On using (36) with $N = \mu\delta$, where δ is a positive constant independent of μ, but dependent upon the closed set S, we see that (34) follows.

For our next result we impose a further condition on the kernel K, namely we require that it possess the following property.

3^0. *$K(t_1, \cdots, t_n)$ vanishes outside some finite cube C_M.*

We shall next prove

Lemma 2. *Let $K(t)$ satisfy properties 1^0, 2^0, and 3^0, let $f(x)$ be a function defined in some domain D, and for some index $p = 0, 1, 2, \cdots$, let $f(x) \in \boldsymbol{k}^{(p)}$ in D, that is let f have continuous derivatives in D of all orders $\leq p$. Then corresponding to any bounded closed set S interior to D there is an index $N(S)$ such that the approximating functions $f_\mu(x)$ defined in (33) have the properties*

$$f_\mu(x) \in \boldsymbol{k}^{(p)} \text{ in } S \qquad \text{for} \qquad \mu > N(S) \tag{39}$$

and

$$\frac{\partial^q f_\mu}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}} \rightrightarrows \frac{\partial^q f}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}} \text{ in } S; \qquad \begin{array}{l} q = 0, 1, \cdots, p \\ q_1 + \cdots + q_n = q \end{array} \tag{40}$$

Let S be any bounded closed set interior to D. Then if μ is sufficiently large all the points

$$\left(x_1 + \frac{t_1}{\mu}, \cdots, x_n + \frac{t_n}{\mu}\right) \tag{41}$$

with x in S and t in C_M will lie in D, indeed it is sufficient that μ be greater than

$$N(S) = \frac{\sqrt{n}\, M}{\text{dist}\,\{S, \text{boundary of } D\}} \tag{42}$$

On using 3^0 we see that (33) can be differentiated under the integral

sign for $\mu > N(S)$ and x in S, yielding for these μ and x

$$(43) \qquad \frac{\partial^q f_\mu}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g\left(x_1 + \frac{t_1}{\mu}, \cdots, x_n + \frac{t_n}{\mu}\right) K(t) dv_t$$

for $q = 0, 1, \cdots, p; q_1 + \cdots + q_n = q$, and with

$$(44) \qquad g(x) = \begin{cases} \dfrac{\partial^q f(x)}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}} & \text{for } x \text{ in } D \\ 0 & \text{elsewhere} \end{cases}$$

Thus (39) holds. The conclusion (40) follows from (43) by Lemma 1 after replacing $g(x)$ in (43) by a (bounded) function $g'(x) = g(x)$ in D' and $= 0$ outside D', where D' is a domain containing S and with closure contained in D.

The two previous results are inadequate for certain purposes since the approximating functions obtained have in general at most the same differentiability properties as the original function f itself. In order to define approximating sequences of functions having stronger differentiability properties we impose a further condition on the kernel namely

4^0. $K(t_1, \cdots, t_n) \varepsilon k^{(r)}$ in $-\infty < t_j < \infty$ for *some* index $r = 0, 1, 2, \cdots$

Kernels satisfying 1^0, 2^0, 3^0 and 4^0 certainly exist. An example is

$$(45) \qquad K(t) = \begin{cases} c^n \Pi_{j=1}^n e^{-\frac{1}{(M-t_j)(M+t_j)}} & \text{for } t \text{ in } C_M \\ 0 & \text{elsewhere} \end{cases}$$

where the constant c is defined by

$$(46) \qquad c^{-1} = \int_{-M}^{M} e^{-\frac{1}{(M-t)(M+t)}} dt$$

It is obvious that this function possesses properties 1^0, 2^0 and 3^0. It can easily be shown that it possesses property 4^0 for *every* index $r = 0, 1, 2, \cdots$. We omit the details.

Approximating functions $f_\mu(x)$ defined as in (33) by means of a kernel $K(t)$ satisfying 1^0, 2^0, 3^0 and 4^0 have in general higher orders of differentiability than the original function $f(x)$. We shall prove

Lemma 3. *Let $K(t)$ satisfy 1^0, 2^0, 3^0 and 4^0, and let $f(x)$ be bounded and measurable in $-\infty < x_j < \infty$, $j = 1, \cdots, n$. Then with*

$f_\mu(x)$ defined as in (33) *we have*

(47) $f_\mu(x) \varepsilon \boldsymbol{k}^{(r)}$ in $-\infty < x_j < \infty, \qquad j = 1, \cdots, n$

For the proof we write

(48) $$f_\mu(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\left(x_1 + \frac{t_1}{\mu}, \cdots, x_n + \frac{t_n}{\mu}\right) K(t) dv_t$$
$$= \mu^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\tau) K[\mu(\tau_1 - x_1), \cdots, \mu(\tau_n - x_n)] dv_\tau$$

Since K vanishes outside a finite cube (by 3^0) (48) shows that for each μ the function $f_\mu(x)$ has the same differentiability properties as the kernel $K(t)$, that is (47) holds for the index r of condition 4^0.

For later reference it is convenient to state explicitly a corollary which we will show to follow from the preceding lemmas.

Corollary 1. *For some index $p = 0, 1, 2, \cdots$, let $\varphi(x_1, \cdots, x_n)$ belong to class $\boldsymbol{k}^{(p)}$ in some domain D. Then there is an approximating sequence $\{\varphi_\mu(x)\}$ such that*

(49) $\varphi_\mu \varepsilon \boldsymbol{k}^{(r)}$ in $-\infty < x_j < \infty$

for every $r = 0, 1, 2, \cdots$

and

(50) $$\frac{\partial^q \varphi_\mu}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}} \rightrightarrows \frac{\partial^q \varphi}{\partial x_1^{q_1} \cdots \partial x_n^{q_n}}$$

in every closed set interior to D, $(q = 0, 1, \cdots, p; q_1 + \cdots + q_n = q)$.

First define

$$f(x) = \begin{cases} \varphi(x) & x \text{ in } D \\ 0 & \text{elsewhere} \end{cases}$$

With a kernel $K(t)$ satisfying 1^0, 2^0, 3^0 and 4^0 (with 4^0 holding for *every* $r = 0, 1, 2, \cdots$, see example (45)) we define $\varphi_\mu(x)$ to be equal to the function $f_\mu(x)$ of (33). Then by Lemma 3, (49) follows and by Lemma 2, (50) follows.

§4. Bounded Functions in Tubes

We consider an arbitrary point set A in real $(x_1, \cdots, x_k)$-space. We do not require that it shall be a domain but we do require that it shall be *rectifiably connected*. By this we understand that any two points P': $(\xi_1, \cdots, \xi_k)$, P'': $(\xi_1'', \cdots, \xi_k'')$ in A shall be connected in A by a point set C: $x_j = \xi_j(s)$, the functions $\xi_j(s)$ being continuous and of bounded variation in s. We now consider the tube T_A in E_{2k}-space, and any domain U containing T_A. If T_A is a domain (that is if A is an open set) then U may be T_A itself, otherwise it has to be

larger. However it need not be a tube. Nevertheless, since U is an open set in E_{2k}, if C is any fixed compact subset of A, and $a > 0$ is given, then there exists an open (k-dimensional) neighborhood of N_a of C, which is not necessarily part of A, such that the domain

$$(51) \qquad U_a\colon \quad (x) \in N_a; \qquad -a < y_j < a, \qquad j = 1, \cdots, k$$

is part of U.

We now consider an analytic function $f(z_1, \cdots, z_k)$ in U, and we assume that it is bounded there, $|f(z)| \leq M$. It is convenient to express $f(z)$ as a function $\varphi(x; y) \equiv f(x_j + iy_j)$ of the $2k$ real components.

For fixed $\lambda > 0$, and fixed $z = (z_1, \cdots, z_k)$ we consider the function

$$(52) \qquad \psi(\xi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(\xi, \eta) e^{\lambda[(z_1-\xi_1-i\eta_1)^2 + \cdots + (z_k-\xi_k-i\eta_k)^2]} dv_\eta$$

where $dv_\eta = d\eta_1 \cdots d\eta_k$, for all (ξ) in A, and we are going to show that it is independent of ξ. In order to prove this we consider the approximating functions

$$(53) \qquad \psi_a(\xi) = \int_{-a}^{a} \cdots \int_{-a}^{a} I(\xi; \eta) dv_\eta$$

where

$$(54) \qquad I(\xi; \eta) \equiv \varphi(\xi; \eta) e^{\lambda[(z_1-\xi_1-i\eta_1)^2 + \cdots + (z_k-\xi_k-i\eta_k)^2]}$$

Let $P'\colon (\xi_1', \cdots, \xi_k')$, $P''\colon (\xi_1'', \cdots, \xi_k'')$ be any two points in A. We must show that $\psi(\xi') = \psi(\xi'')$. For this purpose we take our previous curve $C\colon \xi_j = \xi_j(s)$ joining P' and P'', and the previous sets N_a and U_a, and a fixed bounded neighborhood N of C which contains N_a for $a \geq a_0$. We now consider $\psi_a(\xi)$ in all of N_a and we form the partial derivative $\partial\psi_a(\xi)/\partial\xi_1$ by differentiating (53) under the integral. Now in U_a the integral $I(\xi; \eta)$ is an analytic function of the complex variable $\zeta_1 = \xi_1 + i\eta_1$ and hence $\partial I/\partial\xi_1 = -i\partial I/\partial\eta_1$. Therefore

$$\begin{aligned} i\frac{\partial\psi_a(\xi)}{\partial\xi_1} &= \int_{-a}^{a} \cdots \int_{-a}^{a} \frac{\partial I}{\partial\eta_1} dv_\eta \\ &= \int_{-a}^{a} \cdots \int_{-a}^{a} [I(\xi; a, \eta_2, \cdots, \eta_k) \\ &\qquad - I(\xi; -a, \eta_2, \cdots, \eta_k)] d\eta_2 \cdots d\eta_k \end{aligned}$$

Now the factor $\varphi(\xi; \eta)$ in (54) is in absolute value $\leq M$ in U_a, whereas the exponential factor, for $(\xi) \in N$ and $\eta_1 = \pm a$, is in absolute value

$$\leq K \exp\{-\mu(a^2 + \eta_2^2 + \cdots + \eta_k^2)\} \qquad \text{with} \qquad \mu = \frac{\lambda}{2}$$

From this we deduce, for $(\xi) \varepsilon N$

$$(55) \qquad \left|\frac{\partial\psi_a(\xi)}{\partial\xi_1}\right| \leq L \cdot e^{-\mu a^2}; \qquad \mu = \frac{\lambda}{2}$$

and similarly for $\partial\psi_a(\xi)/\partial\xi_j$. Now, if the functions $\xi_j(s)$ defining the curve C have continuous first derivatives then

$$\psi_a(\xi'') - \psi_a(\xi') = \int_0^1 \left(\frac{\partial\psi_a}{\partial\xi_1}\frac{d\xi_1}{ds} + \cdots + \frac{\partial\psi_a}{\partial\xi_k}\frac{d\xi_k}{ds}\right) ds$$

and hence by (55)

$$\left|\psi_a(\xi'') - \psi_a(\xi')\right| \leq kLe^{-\mu a^2} \int_0^1 \sqrt{\left(\frac{d\xi_1}{ds}\right)^2 + \cdots + \left(\frac{d\xi_k}{ds}\right)^2}\, ds$$
$$= k \cdot L \cdot e^{-\mu a^2} D(P', P'')$$

where $D(P', P'')$ is the length of the curve C. Now, if the curve C does not have continuous tangents, then it can be so approximated in length within N_a, and thus the estimate holds always. Letting $a \to \infty$, we now obtain $\psi(\xi') = \psi(\xi'')$ and thus $\psi(\xi)$ is constant in A.

Therefore we may consider for any $\lambda > 0$ the function

$$(56) \quad f_\lambda(z) = \left(\frac{\lambda}{\pi}\right)^{\frac{k}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(\xi; \eta) e^{\lambda[(z_1-\xi_1-i\eta_1)^2+\cdots+(z_k-\xi_k-i\eta_k)^2]} dv_\eta$$

the expression on the right being independent of ξ, $\xi \varepsilon A$. Now for each λ this function is bounded in any tube T_H of bounded basis. Assume now that our set A is bounded, and that H is its convex hull, and assume that H has an interior, denoting the latter by H_0. Now for $(x_1, \cdots, x_k) \varepsilon A$, put $\xi_j = x_j$ in (56) and thus for $z \varepsilon T_A$, we have

$$(56)' \quad f_\lambda(z) = \left(\frac{\lambda}{\pi}\right)^{\frac{k}{2}} \int_{-\infty}^{\infty} \cdots \int f(x_1 + i\eta_1, \cdots, x_k + i\eta_k)$$
$$e^{-\lambda[(y_1-\eta_1)^2+\cdots+(y_k-\eta_k)^2]} dv_\eta$$

Since $|f(z)| \leq M$ in T_A (even in U) and since the exponential factor in the integrand integrates into $(\pi/\lambda)^{\frac{k}{2}}$ this implies

$$|f(z)| \leq M$$

for all $z \varepsilon T_A$, and all $\lambda > 0$. By Theorem 2 this inequality holds throughout T_H. Now, we can pick a sequence of integers $\{\lambda_1, \lambda_2, \cdots\}$ such that the sequence $\{f_{\lambda_n}(z)\}$ is continuously convergent to

an analytic function $g(z)$ in T_{H^0}. Finally we assume that the neighborhood U of T_A intersects T_{H_0} in a domain. We then claim that $g(z)$ is an analytic continuation of $f(z)$ from U into $U + T_{H_0}$. We will prove this with the aid of the very specialized conclusion stated at the end of section 1. The conclusion will be valid if there exists a point $(x_1^0, \cdots, x_k^0)$ of A which belongs to H_0 or to the boundary of H_0, such that on every compact set of the k-dimensional manifold

$$\operatorname{Re} z_j = x_j^0, \qquad -\infty < \operatorname{Im} z_j < \infty$$
$$f_\lambda(x^0 + iy) \rightrightarrows f(x^0 + iy) \qquad \text{as} \qquad \lambda \to \infty$$

Now this is the place where Lemma 1 comes in. By that lemma applied to (56)′ we see that this takes place for any $(x_1^0, \cdots, x_k^0)$ belonging to A, since $f(z)$ is analytic and bounded on T_A. But since H_0 is the interior of the convex hull of A, then there must be a point x^0 of A either in H_0 or on its boundary. Thus the specialized conclusion stated at the end of section 1 applies and hence $g(z)$ is an analytic continuation of $f(z)$ from U into $U + T_{H_0}$.

The restriction that A be bounded can be removed by making A a limit of an increasing sequence of subsets A, each of which is bounded and rectifiably connected. The corresponding convex sets H_0^n converge toward H_0. Altogether we have the following theorem.

Theorem 3. *If A is any rectifiable connected set in real $(x_1, \cdots, x_k)$-space whose convex hull H has an interior H_0, and if U is any domain in $(z_1, \cdots, z_k)$-space which contains the tube T_A and intersects the tube T_{H_0} in a domain, then any analytic function $f(z)$ which is defined and bounded in U can be continued analytically to a bounded function, with the same bound, into all of $U + T_{H_0}$.*

Remark. If A is a *domain* in E_k and H_0 its convex hull then the analytic continuation of a function from T_A into T_H has already been proved in Theorem 9, Chapter V, and if the original function is bounded in T_A then its continuation into T_{H_0} is again bounded with the same bound. This latter is a consequence of the general property stated in section 1 of Chapter IV.

§5. L_p-NORM FOR VOLUME INTEGRALS

We start from the relation

$$(57) \qquad |f(0)|^p \leq \left(\frac{1}{2\pi}\right)^k \int_0^{2\pi} \cdots \int |f(\rho_j e^{i\alpha_j})|^p dv_\alpha$$

for analytic functions $f(z)$. For $p \geq 1$ this is known to follow from (28) by Hölder's inequality. It actually holds for all $p > 0$ as a

consequence of the deeper relation (18) for $z = 0$, however this does not matter for the present. Assuming that $f(z)$ is analytic in the closed set $A \equiv A_r$: $|z_j| \leq r$, we multiply both sides of (57) by $\rho_1 \cdots \rho_k d\rho_1 \cdots d\rho_k$ and integrate between $0 \leq \rho_j \leq r$. This leads to

$$(58) \qquad |f(0)|^p \leq \frac{1}{\pi^k r^{2k}} \int_A |f(z)|^p dv_x dv_y$$

where $dv_x dv_y \equiv dx_1 dy_1 \cdots dx_k dy_k$ is the $2k$-dimensional volume element in our space E_{2k}.

We now take an arbitrary domain D in E_{2k}, and we introduce, for any $p \geq 1$, the space Λ_p of complex-valued measurable functions $\varphi(x, y)$ for which the norm

$$(59) \qquad ||\varphi|| \equiv ||\varphi||_p = \left(\int_D |\varphi(x, y)|^p dv_x dv_y\right)^{\frac{1}{p}}$$

is finite. This is a Banach space (with complex coefficients), and we emphasize that Λ_p is the total space of such functions in the $2k$ real variables, without regard to analyticity. We now consider in Λ_p the subset L_p consisting of those functions which happen to be analytic, $\varphi(x, y) \equiv f(z)$. If the domain D is too large, for instance if D is the whole E_{2k}, L_p may be empty (but for the element $f \equiv 0$) as the reader will conclude from relation (60) to follow. Or, if it is not empty, it might conceivably be a finite-dimensional vector-space, although no such case is on record. On the other hand, if D has finite volume, then every bounded analytic function in D belongs to L_p, and if D is a bounded domain this includes all polynomials. Now (58) has very striking consequences for the space L_p. Replacing r by r/k, and denoting the constant $\pi^{-k}(r^{-1}k)^{2k}$ by ω_r, we conclude easily that for any point P: $(z_1, \cdots, z_k)$ in D whose Euclidean distance from the boundary of D is $\geq r$, we have

$$(60) \qquad |f(z)| \leq \omega_r^{\frac{1}{p}} ||f||_p$$

Thus collective boundedness of a family $\{f_\alpha(z)\}$ in norm implies collective boundedness in magnitude on every compact subset S of D, and, more precisely, the relation

$$|f_m(z) - f_n(z)| \leq \omega_r^{\frac{1}{p}} ||f_m - f_n||$$

shows that convergence in norm implies continuous convergence in D. Now, if a sequence $\{f_m(z)\}$ in L_p is convergent in norm, then as a sequence in Λ_p it is convergent in norm towards some element $\varphi(x, y)$

of Λ, and a sub-sequence is convergent point-wise, for almost all points, towards $\varphi(x, y)$. On the other hand $\{f_m(z)\}$ is continuously convergent, hence $\varphi(x, y)$ is a uniform limit of analytic functions, and belongs likewise to L_p. Thus L_p is a *closed* linear subspace of Λ_p, and hence a Banach space itself. We now consider any closed linear subspace B_p of L_p, and for a fixed function $\varphi \varepsilon \Lambda_p$ we consider

$$\lambda \equiv \lambda_\varphi = \inf_{g \varepsilon B_p} ||\varphi - g||$$

If $g_n \varepsilon B_p$ is a sequence for which $||\varphi - g_n|| \to \lambda$, then $\{g_n\}$ is bounded in norm. Any sub-sequence contains a continuously convergent part-sequence. If g_0 is the limit of such a part-sequence $\{g_{n_\nu}\}$, then for any compact point set A in D,

$$\int_A |\varphi - g_0|^p dv_x dv_y \leq \lim_{n\to\infty} \int_A |\varphi - g_n|^p dv_x dv_y \leq \lambda^p$$

and hence $||\varphi - g_0|| \leq \lambda$. Thus the inf is attained. We claim that if φ itself is analytic, $\varphi = f \varepsilon L_p$, then the element $g \varepsilon B_p$ for which $||f - g|| = \min$, is unique. In fact, by properties of norm, if g_1 and g_2 are two minimizing functions, then by Minkowski's inequality

$$\left\|f - \frac{g_1 + g_2}{2}\right\| \leq \frac{1}{2}||f - g_1|| + \frac{1}{2}||f - g_2|| = \lambda_f$$

and thus the equality in the former relations must hold, namely

$$\left\|f - \frac{g_1 + g_2}{2}\right\| = \frac{1}{2}||f - g_1|| + \frac{1}{2}||f - g_2||$$

Now, as is well known for Minkowski's inequality, the equality prevails only if for almost all points of D, $(f - g_1)$ and $(f - g_2)$ have a constant real quotient. Now, either $f - g_2 \equiv 0$, in which case $\lambda = 0$ and $g_1 = f = g_2$, or there exists a neighborhood in which $f - g_2 \not\equiv 0$ and $f - g_1 = c(f - g_2)$, and this finally implies that $c = 1$. Thus $g_1 = g_2$, and the minimizing function g is unique.

Now choose a point in D as the origin, and expand all functions $f(z)$ belonging to L_p in power-series about the origin. Consider any collection C of monomials $\{z_1^{n_1} \cdots z_k^{n_k}\}$, and the linear subspace B_p of L_p whose expansions contain no other monomials. This space is closed since convergence in norm implies continuous convergence and hence formal convergence. Now consider the family F of functions of L_p with expansions

$$f(z) \equiv P(z) + (\text{additional terms}),$$

where $P(z)$ is any fixed polynomial with terms other than those of our collection C and the additional terms are members of C. Varying the additional terms means adding an element of B_p. Thus, by the previous result it follows easily that there exists a "minimal" element, that is there is a unique member $f(z)$ of the set F such that $||f|| = \min$. Thus there exist minimal elements of the form

$$
\begin{aligned}
&f_0(z) = 1 + (\text{higher powers}) \\
&f_j(z) = z_j + (\text{higher powers}), \qquad j = 1, \cdots, k
\end{aligned}
\tag{61}
$$

provided, as we assume, that there are any elements of L_p of these particular forms. It was pointed out by Bergman (for $p = 2$) that the $(k + 1)$-minimal functions $f_0, f_1, \cdots, f_k$ of the form (61) have some beautiful properties of invariance. Consider a one-to-one analytic transformation

$$z_j = z_j(z'_1, \cdots, z'_k) \equiv z'_j + (\text{higher powers}) \tag{62}$$

from D onto a domain D', with nonvanishing Jacobian $J(z') = \partial(z_1, \cdots, z_k)/\partial(z'_1, \cdots, z'_k)$. For $p \neq 2$ we will have to assume that $[J(z')]^{\frac{2}{p}}$ is an analytic function on D'. By Theorem 8 of Chapter II, we have

$$\int_D |f(z)|^p dv_x dv_y = \int_{D'} |f(z(z'))[J(z')]^{\frac{2}{p}}|^p dv_{x'} dv_{y'} \tag{63}$$

And it is not hard to see that

$$f(z) \longleftrightarrow f(z(z'))J(z')^{\frac{2}{p}} \tag{64}$$

is a one-to-one transformation of $L_p(z)$ into $L_p(z')$. Also, since $J(z')^{\frac{2}{p}} = 1 +$ (higher powers) it is easy to see that

$$f_j^D(z(z')) \equiv f_j^{D'}(z')J(z')^{-\frac{2}{p}} \qquad j = 0, 1, \cdots, k \tag{65}$$

where $f_j^D(z)$ and $f_j^{D'}(z')$, $j = 0, 1, \cdots, k$, are the minimal functions (61) for D and D' respectively. Hence, at least in the neighborhood of the origin,

$$\frac{f_j^D(z(z'))}{f_0^D(z(z'))} \equiv \frac{f_j^{D'}(z')}{f_0^{D'}(z')}, \qquad j = 1, \cdots, k \tag{66}$$

and thus these quotients are absolute invariants. Omitting the superscript "D," we denote these functions by $w_j(z)$, $j = 1, \cdots, k$ and we see that they have the expansion

$$w_j(z) = z_j + (\text{higher powers}), \qquad j = 1, \cdots, k$$

Now assume that for our domain D, these functions have the special form

$$w_j(z) \equiv z_j \tag{67}$$

and transcribe these functions by (62) into the domain D'. Being invariants, they go over into

$$w'_j(z') \equiv z_j(z'_1, \cdots, z'_k) \equiv z'_j + \text{(higher powers)}$$

Hence, they cannot have the specialized form (67) in the domain D' unless the transformation (62) is the identity $z_j \equiv z'_j$. In other words the specialized form (67) cannot occur for more than one domain D, and thus the specialized form (67) singles out the domain D from all other domains D' which are equivalent with D in the manner described above. Such specialized domains D have been termed "representative" by Bergman.

We may summarize part of these results as follows. Let D be a domain in E_{2k} and for fixed $p > 1$ define the (Banach) space L_p consisting of all functions $f(z)$ analytic and of finite norm in D, $\|f\|_p < \infty$ where the norm is defined in (59). Take some point of D as the origin and consider all functions belonging to L_p in D and having the specialized form $f(z) = 1 +$ (higher powers). If any such functions exist then there is among them a unique one for which $\|f\| = \min.$ We denote it by $f_0(z)$. Similarly for each $j = 1, \cdots, k$ there is a unique function of L_p of the form $f(z) = z_j +$ (higher powers) for which $\|f\| = \min.$ among all such functions in L_p. (This assumes that there is at least one function of this form in L_p.) We denote the unique minimum by $f_j(z)$. With these $k + 1$ functions we define the k functions

$$w_j(z) = \frac{f_j(z)}{f_0(z)}, \qquad j = 1, \cdots, k$$

These k functions are absolute invariants, i.e. if (62) is any one-to-one analytic transformation from D into a domain D', with Jacobian $J(z')$ nonvanishing in D' and with $[J(z')]^{\frac{2}{p}}$ analytic in D', then $w_j^D(z(z')) \equiv w_j^{D'}(z')$, $j = 1, \cdots, k$. All domains D' which are images of D by such transformations are called *equivalent*. Among all equivalent domains there exists at most one domain D for which the functions $w_j(z)$ have the special form (67). Such a specialized domain is the "representative" domain of Bergman.

Bergman has also observed that all bounded domains which are circular (Carathéodory) are representative; in fact not only are the

functions $w_j(z) \equiv z_j$, but $f_0(z) \equiv 1$, and $f_j(z) \equiv z_j$. To see this write

$$f_0(z) = 1 + A_1(z) + A_2(z) + \cdots \tag{68}$$

where $A_n(z)$ is a polynomial of degree n. Then

$$f_0(z_1e^{i\vartheta}, \cdots, z_ke^{i\vartheta}) = 1 + e^{i\vartheta}A_1(z) + e^{2i\vartheta}A_2(z) + \cdots \tag{69}$$

Now, the volume element dv_xdv_y is invariant under the substitution $z_j|z_je^{i\vartheta}$. Hence the function (69) has the same norm as (68). Hence both are minimal, and thus, since they start with the same constant term they must be identical. But $(e^{ni\vartheta} - 1)A_n(z) \equiv 0$ for $0 \leq \vartheta < 2\pi$ implies $A_n(z) \equiv 0$ as claimed. In the case of the function

$$f_j(z) \equiv z_j + A_2(z) + A_3(z) + \cdots \tag{70}$$

we have

$$e^{-i\vartheta}f_j(z_1e^{i\vartheta}, \cdots, z_ke^{i\vartheta}) = z_j + \Sigma_{n=1}^{\infty}e^{in\vartheta}A_{n+1}(z) \tag{71}$$

We again conclude that the functions (70) and (71) must be identical, and this completes the proof of our statement.

§6. Orthogonal Systems

The case $p = 2$ has special features, since L_2 and B_2 are Hilbert spaces. The space L_2 (and B_2) has a complete orthogonal system $(\varphi_0, \varphi_1, \varphi_2, \cdots)$, (which might be finite), with

$$\int \varphi_\alpha\bar{\varphi}_\beta dv = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases} \tag{72}$$

and each element $f \varepsilon \bar{L}_2$ has a unique expansion

$$f \sim \Sigma_{n=0}^{\infty} a_n\varphi_n, \qquad a_n = \int f\bar{\varphi}_n dv \tag{73}$$

the sum converging in norm to f. Also

$$\Sigma_0^{\infty}|a_n|^2 = \int f\bar{f}dv < \infty \tag{74}$$

and every system $\{a_n\}$ with finite sum $\Sigma|a_n|^2$ actually occurs as the coefficients of some function $f \varepsilon L_2$. Since convergence in norm implies continuous convergence, the series (73) is continuously convergent in D. More precisely, by (60)

$$|\Sigma_{n=0}^{N} b_n\varphi_n(z)|^2 \leq \omega_r^2\Sigma_{n=0}^{N}|b_n|^2, \qquad z \varepsilon D^r$$

where D^r is the domain consisting of all points of D whose distance from the boundary of D is $> r$. Putting

$$b_n = \frac{\overline{\varphi_n(z)}}{[|\varphi_0(z)|^2 + \cdots + |\varphi_N(z)|^2]^{\frac{1}{2}}}, \qquad n = 0, 1, \cdots, N$$

for any fixed z in D^r and letting $N \to \infty$, we hence conclude

$$\Sigma_{n=0}^{\infty}|\varphi_n(z)|^2 \leq \omega_r^2, \qquad z \varepsilon D^r \tag{75}$$

Thus for the series (73) we have by Holder's inequality

$$\Sigma_0^{\infty}|a_n\varphi_n(z)| \leq \sqrt{\Sigma_0^{\infty}|a_n|^2 \cdot \Sigma_0^{\infty}|\varphi_n(z)|^2} \leq \omega_r \cdot \sqrt{\Sigma|a_n|^2}, \qquad z \varepsilon D^r$$

and this yields continuous convergence of (73) in D. We now introduce the "kernel"

$$K(z;\bar{\zeta}) \equiv \Sigma_{n=0}^{\infty}\varphi_n(z)\overline{\varphi_n(\zeta)} \tag{76}$$

where (z) and (ζ) are both points of D. The kernel is analytic in (z) and $(\bar{\zeta})$, and the relation (75) reads

$$K(z;\bar{z}) \leq \omega_r^2, \qquad z \varepsilon D^r \tag{77}$$

The kernel (76), and in particular the function $K(z;\bar{z})$ are independent of the special choice of orthogonal system; this follows most easily from properties of Hilbert space. Thus one would expect it to have a geometric interpretation. The most immediate interpretation of a function of pairs of points is that of a geometric "distance." This is true in our case, although somewhat indirectly. It is not easy to interpret $K(z;\bar{z})$ as a "global" distance. However it gives rise to a differential form upon which a Riemannian distance can be based. In fact, if we put

$$g_{\alpha\beta}(z;\bar{z}) = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \log K(z;\bar{z}) \tag{78}$$

then the differential form

$$\Sigma_{\alpha,\beta=1}^{k} g_{\alpha\beta}\, dz_\alpha d\bar{z}_\beta$$

is nonnegative, and what is significant, it is invariant with respect to transformations (62). The invariance follows easily from the fact that under (62), $K(z;\bar{z})$ is multiplied by $J(z)\overline{J(z)}$, and the added factor does not contribute to the matrix (78), since we form the latter from $\log K$, and not K itself.

Also, the function

$$K(z;\bar{z}) = \Sigma_{n=0}^{\infty}|\varphi_n(z)|^2 \tag{79}$$

does not attain a weak maximum in an interior point of D; this can be shown to follow from the results in section 2 and will actually be shown in the next section of the present chapter. Thus, roughly speaking, $K(z;\bar{z})$ attains its maximum (whether finite or infinite) on the boundary. Now, unless the domain D is very pathological, it is

unlikely that K stays bounded in the total domain; at any rate, boundedness in a larger domain $D' \supset D$ would imply the convergence of all series (73) in that larger domain, and so D' would at least have to be an analytic completion of D with regard to the family L_2 in D. Thus we are led to suspect that $K(z; \bar{z})$ increases towards boundary points P at the rate of growth which corresponds to the extent to which the individual points P are obstacles for analytic completion. Bergman has many estimates about the rate of growth in approaching "characteristic manifolds" on the boundary, and for our part we are going to show that conversely, in a Reinhardt domain D containing the origin and having finite volume, the series (79) for $K(z; \bar{z})$ remains convergent in the whole Reinhardt analytic completion of D (see section 8 of Chapter IV). In fact consider any two monomials $\varphi = z_1^{m_1} \cdots z_k^{m_k}$, $\psi = z_1^{n_1} \cdots z_k^{n_k}$ and their inner product

$$(80) \qquad (\varphi, \psi) = \int_D \varphi \bar{\psi} dv_x dv_y \equiv \int_D z_1^{m_1} \bar{z}_1^{n_1} \cdots z_k^{m_k} \bar{z}_k^{n_k} dv_x dv_y$$

Now the volume element $dv_x dv_y$ is invariant under the multicircular transformation $z_j | z_j e^{i\vartheta_j}$, $0 \leq \vartheta_j < 2\pi$ and also this transformation sends D into itself. Thus (80) must be equal to its product with

$$(81) \qquad e^{i(m_1 - n_1)\vartheta_1} \cdots e^{i(m_k - n_k)\vartheta_k}$$

and thus $(\varphi, \psi) = 0$ unless $(m_1 - n_1)^2 + \cdots + (m_k - n_k)^2 = 0$. Hence, on introducing appropriate positive normalizing factors λ, we obtain an orthonormal system of functions in D,

$$(82) \qquad \varphi_{n_1 \cdots n_k} = \lambda_{n_1 \cdots n_k} z_1^{n_1} \cdots z_k^{n_k}; \qquad n_1, \cdots, n_k = 0, 1, 2, \cdots.$$

We are going to show that this system is complete. To this end take any function $f(z_1, \cdots, z_k) \in L_2$ in D and consider its power-series expansion. By Theorem 3 of Chapter II the expansion is valid in the Reinhardt analytic completion D' of D. For convenience we write it in the form

$$(83) \qquad f(z_1, \cdots, z_k) = \Sigma_{\mu_j = 0}^{\infty} a_{\mu_1 \cdots \mu_k} (\lambda_{\mu_1 \cdots \mu_k} z_1^{\mu_1} \cdots z_k^{\mu_k})$$

Now in computing the inner product

$$(84) \qquad (f, \varphi_{n_1 \cdots n_k}) \equiv \int_D f \cdot \bar{\varphi}_{n_1 \cdots n_k} dv_x dv_y$$

we may choose an approximating *Reinhardt domain* D^0 whose closure is contained in D, and approximate to (84) by

$$(85) \qquad \int_{D^0} f \cdot \bar{\varphi}_{n_1 \cdots n_k} dv_x dv_y$$

Now, on the one hand, in D^0 we may replace f by its expansion (83), and on the other hand, the system (82) is orthogonal (though not normalized) in D^0. Thus (85) has the value

$$(86)\qquad |a_{n_1\cdots n_k}|^2\lambda^2_{n_1\cdots n_k}\int_{D^0}|z_1|^{2n_1}\cdots|z_k|^{2n_k}dv_x dv_y$$

and on letting $D^0 \to D$ we see that (84) has the value $|a_{n_1\cdots n_k}|^2$. This leads to the following pair of conclusions: if a function f is orthogonal to all functions (82), then it is identically zero, and thus our system is complete; furthermore the expansion (73), if taken for the system (82), is the expansion (83). Now as we have remarked the latter expansion is valid in the Reinhardt completion D' of D. Hence for any complex numbers $\{a_n\}$ with $\Sigma|a_n|^2 < \infty$ the series (73) is absolutely convergent in D'. Now a well known result of Landau states that if a sequence $\{b_n\}$ has the property that $\Sigma a_n b_n$ converges for every sequence $\{a_n\}$ for which $\Sigma|a_n|^2 < \infty$, then $\Sigma|b_n|^2 < \infty$. Thus in our case the series (79), with the φ's defined in (82), must be convergent in D'.

§7. Surface Integrals

We apply formula (57) to a variable analytic function $f(z;\gamma)$, the parameter γ being an arbitrary element of a space Γ. We now assume on Γ the existence of a measure $d\gamma$, and integrating with respect to this measure we obtain first,

$$\int_\Gamma |f(0;\gamma)|^p d\gamma \leq \left(\frac{1}{2\pi}\right)^k \int_\Gamma d\gamma\left(\int |f(\rho_j e^{i\alpha_j};\gamma)|^p dv_\alpha\right)$$

In the applications to follow it will be possible to interchange the two orders of integration on the right-hand side, as the reader will be able to verify. Thus, on introducing the function

$$(87)\qquad \lambda(z) = \left(\int_\Gamma |f(z;\gamma)|^p d\gamma\right)^{\frac{1}{p}}$$

we obtain the relation

$$(88)\qquad \lambda(0)^p \leq \left(\frac{1}{2\pi}\right)^k \int_0^{2\pi}\cdots\int \lambda(\rho_j e^{i\alpha_j})^p dv_\alpha$$

Thus we can repeat a good number of the conclusions of section 2 for the more general type of function $\lambda(z)$. We observe that the function $\lambda(z)$ is actually more general than $|f(z)|$; in fact $\lambda(z) = |f(z)|$ is the special case of the family $f(z;\gamma)$ being defined for only one point

γ, and the measure of this point being 1. Similarly the "discrete" case, $\lambda(z) = (\Sigma_n|\varphi_n(z)|^p)^{\frac{1}{p}}$ arises by taking the space Γ to consist of the points $\gamma = \{n\}$ with each point having measure 1. Now, if $\lambda(z)$ is defined and has continuous boundary values on the boundary B of a bounded domain D, then in analogy to (25) we have

$$\lambda(z) \leq \sup_{\zeta \varepsilon B} \lambda(\zeta), \qquad z \varepsilon D \tag{89}$$

We can again use the Phragmen-Lindelöf device of multiplying $f(z; \gamma)$ by $\exp [\epsilon(z_1^2 + \cdots + z_k^2)]$ in order to show that (89) also holds if D is a tube T_Δ, in which $\lambda(z)$ is bounded with continuous boundary values, and again the device of multiplying $\lambda(z)$ by $\exp [\alpha_1 z_1 + \cdots + \alpha_k z_k]$ will show that, under the same assumptions, the function

$$\log M(x) = \sup_{-\infty < y_j < \infty} \log \lambda(x_j + iy_j)$$

is a convex function in the basis Δ.

In particular the reader will verify that our results hold, in a tube T_Δ, for the function

$$\lambda(z) = (\textstyle\int_0^{2\pi} \cdots \int |f(z_j + i\gamma_j)|^p d\gamma_1 \cdots d\gamma_k)^{\frac{1}{p}}, \qquad p \geq 1 \tag{90}$$

where $f(z_j) \equiv f(z_1, \cdots, z_k)$ is a bounded and analytic function in T_Δ. As a very special case, assume that $f(z_j)$ has the period $2\pi i$ in each variable. In this case $\lambda(z)$ is dependent only on x, and we obtain (as a generalization of Hardy's theorem) that for any analytic function $g(w_1, \cdots w_k)$ in a Reinhardt region the logarithm of the function

$$(\textstyle\int_0^{2\pi} \cdots \int |g(r_1 e^{i\vartheta_1}, \cdots, r_k e^{i\vartheta_k})|^p d\vartheta_1 \cdots d\vartheta_k)^{\frac{1}{p}}$$

is a convex function of the variables $\log r_1, \cdots, \log r_k$.

Now, take an arbitrary set Γ in the space of the real variables $(\gamma_1, \cdots, \gamma_k)$ and replace (90) by the more general function

$$\lambda_\Gamma(z) = (\textstyle\int_\Gamma |f(z_j + i\gamma_j)|^p d\gamma_1 \cdots d\gamma_k)^{\frac{1}{p}} \tag{91}$$

If our function $f(z)$ is bounded in the tube T_Δ, and if A is a set in Δ whose convex hull H is also contained in Δ, then we obtain first for any bounded set the relation,

$$\lambda_\Gamma(z) \leq \sup \lambda_\Gamma(\zeta)$$

where $z \varepsilon T_H$, $\zeta \varepsilon T_A$. Now, if Γ increases, then so does $\lambda_\Gamma(\zeta)$, and for Γ the total $(\gamma_1, \cdots, \gamma_k)$-space, $\lambda_\Gamma(z)$ depends only on x, that is

$$\lambda(z) \equiv \lambda_p(z) \equiv \lambda_p(x) \equiv (\textstyle\int_{-\infty}^{\infty} \cdots \int |f(x_j + i\gamma_j)|^p d\gamma_1 \cdots d\gamma_k)^{\frac{1}{p}} \tag{92}$$

Letting Γ run over a sequence of cubes which exhaust the total space, we finally obtain

$$\sup_{x \in H} \lambda_p(x) \leq \sup_{\xi \in A} \lambda_p(\xi) \tag{93}$$

(Of course, since $A \subset H$ the equality in (93) actually holds.) In particular we see that $\lambda_p(x)$ is a convex function in all its variables. We have derived this under the assumption which may seem too restrictive at first, that $f(z)$ be bounded. However, it is easy to see that this "restriction" is implied by the assumption that $\lambda_p(x)$ shall be bounded. In fact if Δ is a bounded domain, and $\lambda_p(x) \leq M$ for $x \in \Delta$, then

$$\int_{T_\Delta} |f(z)|^p dv_x dv_y \equiv \int_\Delta dv_x \int_{-\infty}^{\infty} |f(z)|^p dv_y \leq (\text{volume } \Delta) \cdot M^p$$

and once this is established, we conclude from relation (60) that $f(z)$ is bounded in every tube T_A, where A is any compact set in Δ.

In the remainder of this section we will prove a theorem about the boundedness of $\lambda_p(x)$ in which the above "restriction" is substantially relaxed. We will prove it in two steps. First for $f(z)$ itself, and then for $\lambda_p(x)$.

Lemma 4. *If $f(z) \equiv f(z_1, \cdots, z_k)$ is an entire function, if, for all (z)*

$$|f(z)| \leq M e^{A_1|x_1| + \cdots + A_k|x_k|} e^{B_1|y_1| + \cdots + B_k|y_k|} \tag{94}$$

and if

$$|f(iy_1, \cdots, iy_k)| \leq 1, \qquad -\infty < y_j < \infty \tag{95}$$

then

$$|f(z)| \leq e^{A_1|x_1| + \cdots + A_k|x_k|} \tag{96}$$

We will prove relation (96) for

$$x_1 \geq 0, \qquad \cdots, \qquad x_k \geq 0 \tag{97}$$

the other combination of signs then follow immediately. We consider the function

$$g_\epsilon(z) = f(z) e^{-A_1 z_1 - \cdots - A_k z_k} e^{\epsilon(z_1^2 + \cdots + z_k^2)} \tag{98}$$

for $0 < \epsilon < 1$. From (94) we obtain

$$\log |g_\epsilon(z)| \leq \log M + \epsilon(x_1^2 + \cdots + x_k^2) + \Sigma_{j=1}^k (B_j|y_j| - \epsilon y_j^2)$$

and from (95) we obtain

$$\log |g_\epsilon(iy)| \leq 0 \tag{99}$$

We now apply our convexity result of section 2 for

$$M_\epsilon(x) \equiv \log\,[\sup_y |g_\epsilon(z)|]$$

in the octant of the sphere

$$0 \leq \Sigma_{j=1}^k x_j^2 \leq R^2, \qquad x_j \geq 0$$

Since the maximum of $B|y| - \epsilon y^2$ is $B^2/(4\epsilon)$ we see that

$$\begin{aligned} M_\epsilon(x) &\leq \frac{r}{R}\left\{\log M + \epsilon R^2 + \frac{1}{4\epsilon}\,\Sigma B_j^2\right\} \\ &\leq \frac{r}{R}\log M + r\epsilon R + \frac{r}{4}\,\Sigma B_j^2 \cdot \frac{1}{\epsilon R} \end{aligned}$$

where $r^2 = x_1^2 + \cdots + x_k^2$. On making $R = 1/\epsilon$ we obtain

$$M_\epsilon(x) \leq \epsilon r \log M + r + \frac{r}{4}\,\Sigma B_j^2 \tag{100}$$

Letting $\epsilon \to 0$ in (98) and (100) we see that the function

$$g(z) \equiv f(z)e^{-A_1z_1 - \cdots - A_kz_k}$$

is bounded in (97). This now being established, then multiplication by $\exp\,[-\epsilon(z_1 + \cdots + z_k)]$, application of the convexity theorem and a passage to the limit as $\epsilon \to 0$ will show that $|g(z)|$ assumes its maximum for $x_1 = \cdots = x_k = 0$, and since $|g(iy_j)| \leq 1$ we obtain (96).

Theorem 4. *If $f(z)$ is an entire function for which relation* (94) *holds for all z, then the function*

$$\lambda(x) = \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x_j + i\gamma_j)|^p dv_\gamma\right)^{\frac{1}{p}} \tag{101}$$

satisfies the relation

$$\lambda(x) \leq \lambda(0)e^{A_1|x_1| + \cdots + A_k|x_k|} \tag{102}$$

If Γ is a bounded domain in $(\gamma_1, \cdots, \gamma_k)$-space, and if $f(z)$ satisfies relation (94), then the reader will verify that the function $\lambda_\Gamma(z)$ of (91) will satisfy the same relation (94) except with another constant M (depending upon Γ), and a repetition of the proof of Lemma 4 will give the relation

$$\lambda_\Gamma(z) \leq \lambda_\Gamma(0)e^{A_1|x_1| + \cdots + A_k|x_k|} \tag{103}$$

Letting Γ grow into the full space $-\infty < \gamma_j < \infty$, $j = 1, \cdots, k$,

first on the right-hand side of (103) and then on the left-hand side, we obtain relation (102).

§8. Multiple Fourier Integrals

In the present section we will assume that the reader is familiar with the concepts and inversion properties of Fourier transforms in several variables for any L_p-norm, particularly for $p = 2$ (Plancherel).

Theorem 5. *If $f_x(y) \equiv f(z)$ is analytic in a tube T_Δ and if for every compact set A in Δ, the (surface-) L_p-norm $(\int |f_x(y)|^p dv_y)^{\frac{1}{p}}$ is bounded, then for $1 \le p \le 2$ we have*

$$(104) \qquad f(z) \sim \left(\frac{1}{2\pi}\right)^{\frac{k}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(t_1, \cdots, t_k) e^{z_1 t_1 + \cdots + z_k t_k} dv_t$$

where

$$(105) \qquad \varphi(t) \sim \left(\frac{1}{2\pi}\right)^{k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z_1, \cdots, z_k) e^{-t_1 z_1 - \cdots - t_k z_k} dv_y$$

and, for $1/q + 1/p = 1$, we have

$$(106) \qquad (\int |\varphi(t) e^{t_1 x_1 + \cdots + t_k x_k}|^q dv_t)^{\frac{1}{q}} \le (\int |f_x(y)|^p dv_y)^{\frac{1}{p}}$$

For $p = q = 2$, the last relation is an equality,

$$(107) \qquad \int |\varphi(t) e^{t_1 x_1 + \cdots + t_k x_k}|^2 dv_t = \int |f_x(y)|^2 dv_y$$

The full meaning of the inverse relations (104), (105) is as follows. For each (x) there exists a (measurable) transform

$$(108) \qquad \psi(x; t) \sim \left(\frac{1}{2\pi}\right)^{\frac{k}{2}} \int f_x(y) e^{-i(t_1 y_1 + \cdots + t_k y_k)} dv_y$$

for which the inversion formula

$$(109) \qquad f_x(y) \sim \left(\frac{1}{2\pi}\right)^{\frac{k}{2}} \int \psi(x; t) e^{i(y_1 t_1 + \cdots + y_k t_k)} dv_t$$

and the inequality

$$(110) \qquad (\int |\psi(x; t)|^q dv_t)^{\frac{1}{q}} \le (\int |f_x(y)|^p dv_y)^{\frac{1}{p}}$$

hold. Now the theorem claims that there exists a function $\varphi(t)$ such that, for all $x \in \Delta$,

$$(111) \qquad \psi(x; t) e^{-x_1 t_1 - \cdots - x_k t_k} \equiv \varphi(t)$$

By section 4, $f(z)$ is bounded in T_N, where N is a neighborhood of

A. We first assume, in T_N, the growth restriction

$$(112) \qquad |f(z)| \leq Me^{-\epsilon(y_1^2+\cdots+y_k^2)}$$

In this case the right side of (105) is absolutely convergent, and is the limit, as $a \to \infty$, of

$$(113) \quad \varphi_a(x;\, t) \equiv \left(\frac{1}{2\pi}\right)^{\frac{k}{2}} \int_{-a}^{a} \cdots \int_{-a}^{a} f(z_1, \cdots, z_k) e^{-(t_1z_1+\cdots+t_kz_k)} dv_y$$

In order to prove (111) it is enough to show, for each $\alpha = 1, \cdots, k$, that $\partial\varphi_a/\partial x_\alpha$ tends to 0, as $a \to \infty$, uniformly in N. Now, as in the case of the function (53), we see that the derivative of the integral (113) with respect to x_α is essentially the same as with respect to y_α, and the expression representing it contains two boundary terms which on the basis of (112) tend uniformly to 0 as a increases towards ∞. Finally, in order to remove the restriction (112) we form, for $f(z)$ given, the function

$$f_\epsilon(z) = f(z)e^{2\epsilon(z_1^2+\cdots+z_k^2)}$$

It satisfies (112) in T_N, and for its transform we thus have $\psi_\epsilon(x;\, t)$ $e^{-(x_1t_1+\cdots+x_kt_k)} \equiv \varphi_\epsilon(t)$. Now, by (110), since $f_\epsilon(x;\, y)$ converges in L_p-norm to $f(x;\, y)$, then $\psi_\epsilon(x;\, t)$, for each x, converges in L_q-norm to $\psi(x;\, t)$. This implies the convergence in mean, over every finite set, of $\varphi_\epsilon(t)$ to a function $\varphi(t)$, and (111) follows.

As a comment on Theorem 5, we start from any measurable function $\varphi(t)$, and for any $q \geq 1$ (and not just for $q \geq 2$ as in the theorem) we consider the point set H_q for which

$$\mu_q(x) \equiv \left(\int \left|\varphi(t)e^{x_1t_1+\cdots+x_kt_k}\right|^q dv_t\right)^{\frac{1}{q}}$$

is bounded. If x' and x'' are any two points of H_q, and $x_j = \theta x_j' + (1 - \theta)x_j''$, $0 \leq \theta \leq 1$, is a point of the conjoining segment, then by Hölder's inequality, we easily obtain $\mu_q(x) \leq (\mu_q(x'))^\theta(\mu_q(x''))^{1-\theta}$. Thus $\log \mu_q(x)$ is a convex function, and H_q is a convex set. We assume that H_q has an interior, which we denote by H_q^0, and we note that $\mu_q(x)$ is bounded on every compact set of H_q^0. Take any point of H_q^0, say the origin. There exist a $\delta > 0$ and an M such that

$$\int |\varphi(t)|^q e^{q(t_1x_1+\cdots+t_kx_k)} dv_t \leq M$$

for $|x_\alpha| < \delta$. On substituting for $(x_1, \cdots, x_k)$ the points $(\pm\delta, \cdots, \pm\delta)$ an adding up over all combinations of signs we obtain

$$\int |\varphi(t)|^q e^{q\delta(|t_1|+\cdots+|t_k|)} dv_t \leq 2^k M$$

Hence, by Hölder's inequality

$$\int |\varphi(t)| dv_t \leq 2^{\frac{k}{q}} M^{\frac{1}{q}} (\int e^{-p\delta(|t_1|+\cdots+|t_k|)} dv_t)^{\frac{1}{p}}$$

and thus the origin is a point of H_1^0(for $q = 1$). By a slight elaboration of this argument we see:

Remark 1. *The domain* H_q^0 *in which* $\mu_q(x)$ *is finite increases as* q *decreases.*

Another elaboration will give the following result.

Remark 2. *The integral*

$$\int \varphi(t) e^{z_1t_1+\cdots+z_kt_k} dv_t$$

is continuously convergent and defines an analytic function in $T_{H_q^0}$.

In particular we obtain the following:

Supplement to Theorem 5. *The integral in* (104) *is absolutely and continuously convergent, thus* (104) *is an ordinary relation in* T_Δ. However, for $p > 1$, no assumptions on the nature of analyticity of $f(z)$ will guarantee off-hand that (105) is a literal relation as is known for $p = 1$.

As a counterpart to Remark 1 we observe that whenever $\int |f(y)|^p dv_y \leq M$ and $|f(y)| \leq N$, then, for $1 \leq p \leq 2$, $\int |f(y)|^2 dv_y \leq N^{2-p} \cdot M$, thus, in Theorem 5, as long as we do not approach points (x) on the boundary, there is not much loss of generality in assuming $p = 2$. Furthermore relation (107) is always present, even if $p \leq 2$.

§9. Functions of Integrable Square

In the case of $p = q = 2$, the situation is exceedingly simple. A function $f_x(y) \equiv f(z)$ in a tube T_Δ is analytic, and the squared norm

$$\tilde{\lambda}(x)^2 = \int \cdots \int |f_x(y)|^2 dv_y \tag{114}$$

is bounded in every compact set in Δ if and only if there exists a measurable function $\Phi(t)$ for which the squared norm

$$\lambda(x)^2 = \int \cdots \int |\Phi(t)|^2 e^{2(x_1t_1+\cdots+x_kt_k)} dv_t \tag{115}$$

is bounded and such that

$$f(z) = \left(\frac{1}{2\pi}\right)^k \int \cdots \int \Phi(t) e^{z_1t_1+\cdots+z_kt_k} dv_t \tag{116}$$

Also the two norms are equal, $\lambda(x) \equiv \tilde{\lambda}(x)$

Now, assume that Δ contains a collection of rays $x_\alpha \equiv \rho\xi_\alpha$, $0 < \rho < \infty$, with

$$\xi_1^2 + \cdots + \xi_k^2 = 1 \tag{117}$$

The origin $x_\alpha = 0$ may be on the boundary of Δ but the norm $\lambda(0)$ as defined in (115) shall be finite. Suppose that for some (ξ) in (117) there exists a real number $h(\xi)$ such that the function $\Phi(t)$ vanishes (almost everywhere) outside the half space

$$H(\xi): \qquad t_1\xi_1 + \cdots + t_k\xi_k \leq h(\xi)$$

In this case, for $x_\alpha = \rho\xi_\alpha$, $0 < \rho < \infty$, we have

$$\lambda(x)^2 \leq e^{2h(\xi)} \int_{H(\xi)} |\Phi(t)|^2 dv_t \leq e^{2\rho h(\xi)} \cdot \lambda(0)^2$$

and hence

$$\overline{\lim}_{\rho\to\infty} \frac{\log \lambda(\rho\xi_\alpha)}{\rho} \leq h(\xi)$$

Conversely, if for some $s(\xi)$, the function $\Phi(t)$ is different from 0 on a set Γ of positive measure in the half-space $t_1\xi_1 + \cdots + t_k\xi_k > s(\xi)$, then

$$\lambda(x)^2 \geq e^{2\rho s(\xi)} \int_\Gamma |\Phi(t)|^2 dv_t$$

Altogether, we see: *if $h(\xi)$ is the smallest real number such $\Phi(t)$ vanishes outside the halfspace $H(\xi)$, then*

$$\lim_{\rho\to\infty} \frac{\log \lambda(\rho\xi_\alpha)}{\rho} = h(\xi) \tag{118}$$

the limit on the left existing. It can be $-\infty$ only if $f(z) \equiv 0$, and $+\infty$ *only if no genuine half-space $H(\xi)$ exists.*

On the basis of Theorem 4 we now obtain.

Theorem 6. *If $f(z_1, \cdots, z_k)$ is an entire function for which*

$$|f(z)| < Me^{A(|z_1|+\cdots+|z_k|)} \tag{119}$$

and if $f(iy_1, \cdots, iy_k)$ is of integrable square in $-\infty < y_j < \infty$, $j = 1, \cdots, k$, then

$$f(z) = \left(\frac{1}{2\pi}\right)^k \int_{-A}^{A} \cdots \int_{-A}^{A} \psi(t) e^{z_1t_1+\cdots+z_kt_k} dv_t$$

where $\psi(t)$ is of integrable square. The smallest convex domain in t-space outside of which $\psi(t)$ vanishes almost everywhere is the intersection of all half-planes $t_1\xi_1 + \cdots + t_k\xi_k \leq h(\xi)$ where $h(\xi)$ is defined by (118), *the limit in* (118) *existing.*

Another interesting case arises if Δ is a convex "cone" through the origin and $\lambda(x)$ is bounded in Δ. In this case $h(\xi) \leq 0$, and thus $\Phi(t)$ vanishes outside the cone "dual" to Δ. It is enveloped by the

hyper-planes through the origin which are perpendicular to the generators of Δ. For instance if Δ is the octant

$$x_\alpha > 0, \qquad \alpha = 1, \cdots, k \tag{120}$$

then the dual is the octant $t_\alpha < 0$ and it is easy to conclude that $f(z)$ is analytic, and $\int |f_x(y)|^2 dv_y$ is bounded in the octant (120) if and only if $f(z)$ is representable in this octant by

$$f(z) = \left(\frac{1}{2\pi}\right)^{\frac{k}{2}} \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \Phi(t) e^{z_1 t_1 + \cdots + z_k t_k} dv_t$$

with

$$\int_{-\infty}^{0} \cdots \int_{-\infty}^{0} |\Phi(t)|^2 dv_t < \infty$$

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Metrics of the type of Bergman's (see section 5) can be introduced from different standpoints, and there are methods of constructing them either (1) on certain types of domains in $(z_1, \cdots, z_k)$-space, or (2) on certain types of complex analytic spaces in the large, in particular closed (compact) spaces of certain properties. In this connection see:

13. H. CARTAN, Sur les domaines bornés homogenès de l'espace de n variables complexes, *Hamburg Univ. Math. sem. Abhandl.*, Vol. 11 (1936), pp. 106–162.
14. C. L. SIEGEL, Symplectic geometry, *Amer. J. Math.*, Vol. 45 (1943), pp. 1–86.
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In the final section of No. 13 an application of Bergman's metric to so-called symmetric domains is given and in Nos. 13–17 the construction and properties of certain "hyperbolic" geometries on domains of automorphic functions is discussed. In No. 17 there is also a systematic introduction to a familiar type of elliptic geometry on the closed projective space in several complex variables, and in No. 18 there is a brief comment on the contrast between hyperbolic geometries in bounded open domains and elliptic geometries on certain types of closed domains. We also single out for reference the book:

19. W. F. D. HODGE, *Theory and applications of harmonic integrals*, Cambridge, 1941.

In Chapter IV of this book a metric is introduced into algebraic varieties, and then the author reviews many known properties of algebraic varieties in the light of such a metric. A close analysis of

the work then will show that many properties Hodge derives stem exclusively from his metric on a compact complex space, the algebraic origin of the variety being irrelevant.

A somewhat different approach to constructing a metric in complex domains has been made by:

20. C. CARATHÉODORY, Über die Abbildungen, die durch Systeme von analytische Funktionen von mehreren Veränderlichen erzeugt werden, *Math. Zeits.*, Vol. 34 (1932), pp. 758–792.

However, Carathéodory's metric has some recalcitrant features and seems less promising for further development.

With regard to the topic of our section 8, in the symplectic geometry of C. L. Siegel there occurs a significant "octant" which is different from the one given in the text, and it is also "self-dual." For additional details see No. 6. For a problem involving octants in Fourier analysis see also:

21. S. BOCHNER, Boundary values of analytic functions in several variables and of almost periodic functions, *Annals of Math.*, Vol. 45 (1944), pp. 708–722.

No. 21 is in connection with:

22. S. BERGMAN and J. MARCINKIEWICZ, Sur les fonctions analytiques de deux variables complexes, *Fundamenta Matemat.*, Vol. 33 (1939), pp. 75–94.

CHAPTER VII

The Theory of Hartogs. Subharmonic Functions

§1. Harnack's Theorem

In addition to using the fundamental inequality originally due to Jensen, Hartogs also employs the theorem of Harnack. The latter states that if a sequence of (real) harmonic functions which are collectively bounded from above are monotonely decreasing, then either they converge uniformly to a harmonic function proper, or they diverge "uniformly" to $-\infty$. Hartogs noticed that the theorem also works for the larger class of "subharmonic functions," and this incidentally laid the foundation for the study of such functions.

We will consider the extended concept of Lebesque integral as explained in section I, Chapter VI. We consider (real-valued) functions of a variable ϑ, where ϑ is a point on a space T with Lebesgue measure, *the total space having finite measure.* For convenience we will use the symbol $d\vartheta$ under integrals to indicate integration with respect to the measure given. For any finite real number c we denote by L_c the class of functions $\{u(\vartheta)\}$ which are defined at every point of and measurable on T, and for which

$$(1) \qquad -\infty \leq u(\vartheta) \leq c$$

Lemma 1. *If a sequence $\{u_n(\vartheta)\}$ belongs to some L_c, then*

$$(2) \qquad \overline{\lim}_{n\to\infty} \int u_n(\vartheta)d\vartheta \leq \int \overline{\lim}\, u_n(\vartheta)d\vartheta$$

In particular, if $u_n \nearrow u \leq c$ then

$$(3) \qquad \lim_{n\to\infty} \int u_n(\vartheta)d\vartheta \leq \int u(\vartheta)d\vartheta$$

and if $c \geq u_n \searrow u$, then

$$(4) \qquad \lim_{n\to\infty} \int u_n(\vartheta)d\vartheta = \int u(\vartheta)d\vartheta$$

This lemma simply states conveniently a property of our integral, and we accept it without investigation.

In addition to the point set T we consider any point set D whose general point will be denoted by z. Consider a function $P(z, \vartheta)$ on

the product space (D, T) with the following properties (generalized Poisson kernel)

(i) for each z in D, $P(z, \vartheta)$ is measurable in ϑ,

(ii) $\int_T P(z, \vartheta)d\vartheta = 1, \qquad z \in D$,

(iii) there exist finite functions $a(z)$, $A(z)$ in D, such that

$$0 < a(z) \leq P(z, \vartheta) \leq A(z) < \infty \tag{5}$$

for (z, ϑ) in (D, T).

We can now state

Theorem 1 (Harnack). *If $\varphi_n(\vartheta) \in L_c$, $n = 1, 2, \cdots$, and if*

$$\int \limsup_n \varphi_n(\vartheta)d\vartheta > -\infty \tag{6}$$

then there exists a sequence of numbers $\epsilon_n \searrow 0$ such that for any kernel $P(z, \vartheta)$ possessing properties (i) (ii) and (iii) and for any sequence $\{\psi_n(z)\}$ the relations

$$\psi_n(z) \leq \int P(z, \vartheta)\varphi_n(\vartheta)d\vartheta, \qquad n = 1, 2, \cdots \tag{7}$$

imply the relations

$$\psi_n(z) \leq \int P(z, \vartheta)[\limsup_\mu \varphi_\mu(\vartheta)]d\vartheta + \epsilon_n A(z) \tag{8}$$

for $n = 1, 2, \cdots$.

If however

$$\int \limsup_n \varphi_n(\vartheta)d\vartheta = -\infty \tag{9}$$

then there exist a sequence $\epsilon_n \searrow 0$ and an index N such that (7) *implies*

$$\psi_n(z) \leq -a(z)\frac{1}{\epsilon_n} + c; \qquad n \geq N$$

In fact, putting

$$\varphi_n'(\vartheta) = \sup_m \{\varphi_{n+m}(\vartheta)\}$$
$$\varphi_0(\vartheta) = \overline{\lim}_{n\to\infty} \varphi_n(\vartheta)$$

we have

$$\varphi_n(\vartheta) \leq \varphi_n'(\vartheta) \leq c$$
$$\varphi_n'(\vartheta) \searrow \varphi_0(\vartheta)$$
$$\psi_n(z) \leq \int P(z, \vartheta)\varphi_n'(\vartheta)d\vartheta$$

and hence

$$\psi_n(z) \leq \int_T P(z, \vartheta)\varphi_0(\vartheta)d\vartheta + A(z)\int_T(\varphi_n'(\vartheta) - \varphi_0(\vartheta))d\vartheta$$

Now, if (6) holds, then by ordinary Lebesque theory

$$\int_T(\varphi_n'(\vartheta) - \varphi_0(\vartheta))d\vartheta \searrow 0$$

and this proves (8). If however (9) holds, then

$$\int \varphi_n'(\vartheta)d\vartheta \searrow -\infty,$$

and hence

$$\begin{aligned}\psi_n(z) &\leq \int P(z, \vartheta)c\,d\vartheta + \int P(z, \vartheta)(\varphi_n'(\vartheta) - c)d\vartheta \\ &\leq c - a(z)\int(c - \varphi_n'(\vartheta))d\vartheta\end{aligned}$$

and this completes the proof of the lemma.

§2. Hartogs' Main Theorem

We will cast the principal reasoning in the shape of a lemma.

Theorem 2. *If D is a domain in complex $(z_1, \cdots, z_k)$-space, if $\{f_{n_1\cdots n_l}(z)\}$ is a multi-sequence of analytic functions in D, if p_λ, P_λ $(0 < p_\lambda < P_\lambda \leq \infty)$ are two sets of radii, $\lambda = 1, \cdots, l$, if the power-series*

$$\Sigma_{n_1=0}^{\infty} \cdots \Sigma_{n_l=0}^{\infty} f_{n_1\cdots n_l}(z) w_1^{n_1} \cdots w_l^{n_l} \tag{10}$$

is known to be absolutely-continuously convergent in

$$[z \in D, \quad |w_\lambda| < p_\lambda] \tag{11}$$

and absolutely convergent in

$$[z \in D, \quad |w_\lambda| < P_\lambda] \tag{12}$$

then it is also absolutely continuously convergent in (12).

It is sufficient to establish the conclusion in some neighborhood D_0 of an arbitrary point (ζ^0) in D. For convenience let (ζ^0) be the origin, let (ρ) be such that

$$\bar{D}_0: \qquad |z_j| \leq \rho_j \tag{13}$$

is contained in D, let D_0 be the interior of (13), let $T = \{\vartheta_1, \cdots, \vartheta_k\}$ be the k-dimensional set

$$\zeta_j = \rho_j e^{i\vartheta_j}, \qquad 0 \leq \vartheta_j < 2\pi$$

and as in Chapter VI let

$$P(z, \vartheta) = \Pi_{j=1}^{k} \frac{1}{2\pi} \frac{\rho_j^2 - r_j^2}{\rho_j^2 - 2\rho_j r_j \cos(\varphi_j - \vartheta_j) + r_j^2} \tag{14}$$

for $z_j = r_j e^{i\varphi_j}$. As shown in section 1, Chapter VI, we have inequality

(5), and for $0 < r_j^0 < \rho_j$, we have

$$0 < a_0 \leq a(z)_0 \leq A(z) \leq A_0$$

uniformly in $|z_j| \leq r_j^0$.

We now take any radii $0 < p'_\lambda < P'_\lambda < \infty$, such that $p'_\lambda < p_\lambda$, $P'_\lambda < P_\lambda$ and we denote the largest of the quotients P'_λ / p'_λ by q. The assumptions of our lemma imply

$$\lim_{n_1+\cdots+n_l\to\infty} |f_{n_1\cdots n_l}(z)| P_1'^{n_1} \cdots P_l'^{n_l} = 0$$
$$|f_{n_1\cdots n_l}(z) p_1'^{n_1} \cdots p_l'^{n_l}| \leq M_0, \qquad z \in \bar{D}_0$$

Hence the functions

$$\varphi_{n_1\cdots n_l}(\vartheta) = \frac{1}{n_1 + \cdots + n_l} \log \left| \frac{f_{n_1\cdots n_l}(\rho_j e^{i\vartheta_j}) P_1'^{n_1} \cdots P_l'^{n_l}}{M_0} \right| \tag{15}$$

have the following two properties:

$$\varphi_{n_1\cdots n_l}(\vartheta) \leq c (= \log q)$$
$$\limsup_{n_1+\cdots+n_l\to\infty} \varphi_{n_1\cdots n_l}(\vartheta) \leq 0$$

We now apply Theorem 1 of Chapter VI in conjunction with Theorem 1 of the present chapter, breaking into two cases according as

$$\int \limsup \varphi_{n_1\cdots n_l}(\vartheta) d\vartheta_1 \cdots d\vartheta_l$$

is finite or $-\infty$. If the limit superior is finite then we may discard the integral in the right member of (8), its value being ≤ 0, and since $\epsilon_n A_0 \searrow 0$ as $\epsilon_n \searrow 0$ we conclude that corresponding to any positive number ϵ there exists an integer N such that, for $n_1 + \cdots + n_l \geq N$ and for $|z_j| \leq r_j^0$, we have

$$|f_{n_1\cdots n_l}(z) P_1'^{n_1} \cdots P_l'^{n_l}| \leq M_0 e^{\epsilon(n_1+\cdots+n_l)} \tag{16}$$

If, on the other hand, the limit superior in question is $-\infty$, then on using the second half of Theorem 1 and the fact that $-a_0/\epsilon_n + \log q$ decreases to $-\infty$ as $\epsilon_n \searrow 0$ we conclude a fortiori our inequality (16) for $|z_j| \leq r_j^0$. Since in both cases ϵ can be an arbitrarily small positive number this implies that the series (10) is absolutely continuously convergent in $[|z_j| < r_j^0, |w_\lambda| < P'_\lambda]$, and since P'_λ can be chosen arbitrarily near P_λ (arbitrarily large if $P_\lambda = +\infty$) this completes the proof of our lemma.

The significant conclusion from the lemma, which the reader will easily draw for himself, is embodied in the following theorem.

Theorem 2. (Hartogs' lemma). *If $f(z_1, \cdots, z_k; w_1, \cdots, w_l)$ is analytic in the topological product*

$$[z \in D; |w_\lambda| < p_\lambda]$$

and if for each $z \in D$, $f(z; w)$ is analytic in $[|w_\lambda| < P_\lambda]$, *then $f(z; w)$ is analytic in the product*

$$[z \in D; |w_\lambda| < P_\lambda]$$

§3. Contributions of Osgood

Lemma 3. *If a function $f(z_1, \cdots, z_k)$ in a domain D of E_{2k} is analytic in each variable z_j and if it is bounded in all variables, then it is analytic in all variables.*

By Theorem 2, Chapter II, it is sufficient to prove that $f(z)$ is continuous in D. In any polycylindrical neighborhood in D, we consider $f(z)$ as a family of functions in one variable z_j, the other variables $z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_k$ being parameters by which to designate members of the family. By section 5, Chapter II, since $f(z)$ is bounded, our family is majorized, and hence their derivatives $\partial f/\partial z_j$ are likewise majorized. Thus the partial derivatives $\partial f/\partial z_1, \cdots, \partial f/\partial z_k$ are bounded in the neighborhood of each point. Viewing $f(z)$ as a function in the real variables $x_1, y_1, \cdots, x_k, y_k$ we see that its $2k$ partial derivatives of first order are locally bounded, and this in itself implies continuity of $f(z)$.

Theorem 3. *If $A = \{\alpha\}$, $B = \{\beta\}$ are bounded sets in Euclidean spaces of any dimensions, A closed and B open, and if the nonnegative function $\lambda(\alpha, \beta)$ in (A, B) is continuous in each variable, then there exists an open subset B' of B such that $\lambda(\alpha, \beta)$ is bounded in (A, B').*

The function

$$\mu(\beta) = \sup_{\alpha \in A} \lambda(\alpha, \beta)$$

has a finite value for each $\beta \in B$. Also, for each n, the set B_n: $[\mu(\beta) \leq n]$ is a relatively closed subset of B. In fact, if $\beta_p \to \beta_o$ and $\mu(\beta_p) \leq n$, then $\lambda(\alpha, \beta_p) \leq n$, and hence by continuity $\lambda(\alpha, \beta_o) \leq n$, and this implies $\mu(\beta_o) \leq n$. Now, if the theorem were false, then no B_n could contain an interior, and the open sets $B - B_n$ would be everywhere dense in B. This would permit us to construct, by induction on n, a sequence of sets $C_1 \supset C_2 \supset \cdots$ such that each C_n were the closure of an open set, and $\mu(\beta) > n$ for $\beta \in C_n$. The sets C_n would contain a common point $\bar{\beta}$, and at this point $\mu(\beta)$ could not be finite.

§4. Hartogs' Theorem on Analyticity in Each Variable

Theorem 4. *Let $f(z_1, \cdots, z_k)$ be defined in a domain D in E_{2k} and let f have the property that for every point $(a_1, \cdots, a_k)$ of D each of the functions*

$$f(a_1, \cdots, a_{j-1}, z_j, a_{j+1}, \cdots, a_k), \qquad j = 1, \cdots, k$$

is analytic in the single variable z_j in the neighborhood of a_j. Then f is analytic in $(z_1, \cdots, z_k)$ in D.

We assume that $(0, \cdots, 0)$ is a point of D, and we shall prove that f is analytic in a neighborhood of $(z) = (0)$. If

$$|z_1| < U_1, \qquad \cdots, \qquad |z_k| < U_k \tag{17}$$

is a neighborhood of (0) whose closure is contained in D, and if we identify the point set A of Theorem 3 with $|z_1| \leq U_1$, and the point set B with

$$|z_2| < \frac{U_2}{3}, \qquad \cdots, \qquad |z_k| < \frac{U_k}{3} \tag{18}$$

then we conclude the existence of a subneighborhood

$$|z_2 - z_2'| < \alpha_2, \qquad \cdots, \qquad |z_k - z_k'| < \alpha_k \tag{19}$$

of (18), such that f is bounded, and hence by Lemma 3, also analytic in the product of (19) with $|z_1| < U_1$. We will now apply Theorem 2 to the following situation: $f(z_1, \cdots, z_k)$ is analytic in

$$|z_1| < U_1, \qquad |z_2 - z_2'| < \alpha_2, \qquad \cdots, \qquad |z_k - z_k'| < \alpha_k$$

and for each $(z_1, z_3, \cdots, z_k)$ in

$$|z_1| < U_1, \qquad |z_3 - z_3'| < \alpha_3, \qquad \cdots, \qquad |z_k - z_k'| < \alpha_k$$

it is analytic in z_2 for $|z_2| < U_2$. By Theorem 2 we may now envisage the largest circle which is both concentric with $|z_2 - z_2'| < \alpha_2$ and entirely included in $|z_2| < U_2$. Since $|z_2'| < U_2/3$, this largest circle includes the circle $|z_2| < U_2/3$, and thus we conclude that f is analytic in all variables in the point set

$$|z_1| < U_1, \qquad |z_2| < \tfrac{1}{3}U_2, \qquad |z_3 - z_3'| < \alpha_3, \qquad \cdots, \qquad |z_k - z_k'| < \alpha_k$$

Starting again in the same fashion we now concentrate on the fact that for each $z_1, z_2, z_4, \cdots, z_k$, f is analytic in z_3, in the larger domain $|z_3| < U_3/3$, and we again obtain that f is analytic in all variables in

$|z_1| < U_1, \quad |z_2| < \frac{1}{3}U_2, \quad |z_3| < \frac{1}{3}U_3, \quad |z_4 - z_4'| < \alpha_4, \quad \cdots$
Continuing in this way we conclude that $f(z)$ is analytic in

$$|z_1| < U_1, \quad |z_2| < \tfrac{1}{3}U_2, \quad \cdots, \quad |z_k| < \tfrac{1}{3}U_k$$

and so $f(z)$ is, in particular, analytic in the neighborhood of the origin.

§5. Extension of Hartogs' Main Theorem

We consider two sets of variables $(z_1, \cdots, z_k)$, $(w_1, \cdots, w_l)$, the first set complex, the other variables complex or real. As in Chapter IV a domain D in (z, w)-space will be presented as a union of intersections relative to one set of variables. However this time the basis of D will be in the z-space, thus

$$D = [z \varepsilon B;\ w \varepsilon \Delta(z)] \tag{20}$$

We consider a larger domain over the same basis,

$$\tilde{D} = [z \varepsilon B;\ w \varepsilon \tilde{\Delta}(z)] \tag{21}$$

and we assume that every connected component of $\tilde{\Delta}(z)$ contains a nonempty component of $\Delta(z)$.

Theorem 5. (*Generalized Hartogs' lemma.*) *If $f(z, w)$ is analytic in (z, w) in D and, for each z in B, is analytic in $\tilde{\Delta}(z)$, then $f(z, w)$ is analytic in $\tilde{D}$.*

We first assume w also complex. If, for fixed $z = z'$, a rectifiable Jordan curve $w = w(\vartheta)$, $0 \le \vartheta \le 1$, in $\tilde{\Delta}(z)$ joins a point $w^0 = w(0)$ in $\Delta(z)$ with a point $w' = w(1)$ in $\tilde{\Delta}(z)$ then there exists an $\epsilon > 0$ such that for $\vartheta_* = 0$ and other sufficiently small ϑ_*, the point set

$$|z - z'| < \epsilon, \qquad |w - w(\vartheta_*)| < \epsilon \tag{22}$$

lies in D, and for all $\vartheta = \vartheta^*(0 \le \vartheta^* \le 1)$, the point set

$$|z - z'| < \epsilon, \qquad |w - w(\vartheta^*)| < \epsilon \tag{23}$$

lies in $\tilde{D}$. Thus $f(z, w)$ is analytic in (z, w) in (22) and analytic in w in (23). For the proof of the theorem it is sufficient to show that in a finite number of steps, ϑ_* can be successively increased so as to reach $\vartheta_* = 1$. Given ϑ_* we pick a value $\vartheta^* > \vartheta_*$ such that $\frac{\epsilon}{2} < |w(\vartheta^*) - w(\vartheta_*)| < \epsilon$. Then in addition to being analytic in w in (23) our function is analytic in (z, w) in

$$|z - z'| < \epsilon, \qquad |w - w(\vartheta^*)| < \text{some small number}$$

By Theorem 4, $f(z, w)$ is analytic in (z, w) in (23). Thus ϑ_* has been

replaced by the larger number ϑ^*, and owing to the rectifiability of $w(\vartheta)$, the point $\vartheta_* = 1$ itself is thus within reach.

The case of real variables w will be treated very briefly. We take an $\epsilon > 0$ such that 1°) $f(z, w)$ is analytic in (z, w) for $|z - z'| < \epsilon$, w complex and $|w - w(\vartheta_*)| < \epsilon$ and 2°) $f(z, w)$ is analytic in w for $|z - z'| < \epsilon$ and w complex and $|w - w(\vartheta^*)| < \epsilon$. The remainder of the proof will follow.

Given any open sets B, $\Delta^+(z)$ respectively, then the set

$$[z \varepsilon B;\ w \varepsilon \Delta^+(z)] \tag{24}$$

need not be an open set in (z, w)-space. In fact for any $z^0 \varepsilon B$ the layer $[z = z^0;\ w \varepsilon \Delta^+(z^0)]$ may protrude out of the interior of the set (24). In order to trim (24) of these protuberances we replace $\Delta^+(z)$ by the set

$$\Delta(z) = \lim \inf_{(z') \to (z)} \Delta^+(z') \tag{25}$$

It is defined as the union of all (sufficiently small) open sets S, each of which is common to all point sets $\Delta^+(z')$, for $|z' - z| < \epsilon(S)$. The reader will have no great difficulties in proving that

$$[z \varepsilon B;\ w \varepsilon \Delta(z)] \tag{26}$$

is the largest open set in (24) and that the following theorem holds.

Theorem 6. (*Rewording of Theorem* 5.) *If $f(z, w)$, with all z_j complex, is analytic in (z, w) in a domain $[z \varepsilon B;\ w \varepsilon \Delta^-(z)]$, and if for each z, $f(z, w)$ can be connectedly continued from $\Delta^-(z)$ into a larger open set $\Delta^+(z)$, then $f(z, w)$ is analytic in (z, w) in the domain* (26), *where $\Delta(z)$ is defined by* (25).

§6. Hartogs Domains and Subharmonic Functions

We consider an analytic function

$$f(z;\ w) = \Sigma_{n=0}^{\infty} f_n(z_1, \cdots, z_k) w^n \tag{27}$$

of the $k + 1$ complex variables $z_1, \cdots, z_k, w$ in a Hartogs domain

$$[z \varepsilon B;\ |w| < \sigma^-(z)] \tag{28}$$

For each z, the power-series (27) has a radius of convergence $\sigma^+(z) \geq \sigma^-(z)$, and if we introduce the radius

$$\sigma(z) = \lim \inf_{z' \to z} \sigma^+(z') \tag{29}$$

then by Theorem 6 the Hartogs domain

$$z \varepsilon B; \qquad |w| < \sigma(z) \tag{30}$$

is *the* largest Hartogs domain with the given basis B into which $f(z; w)$ can be continued as function in (z, w). Holding (28) fast, we consider $\sigma(z)$ for all $f(z; w)$ analytic in (28), and we form the functions

$$\begin{aligned} \tilde{\sigma}_0(z) &= \inf_f \sigma(z) \\ \tilde{\sigma}(z) &= \liminf_{z' \to z} \tilde{\sigma}_0(z') \end{aligned}$$

Obviously the domain

$$z \in B; \qquad |w| < \tilde{\sigma}(z) \tag{31}$$

is the Hartogs completion of (28); it is *the* maximal Hartogs domain (see Chapter IV) into which all $f(z, w)$ can be continued simultaneously from (28).

The radii $\sigma(z)$ and $\tilde{\sigma}(z)$ of largest or maximal Hartogs domains possess certain important characteristics. In sections 7, 8 and 9 we shall consider certain other types of domains and the functions defining the largest or maximal domains of these types. In each case we will find that these functions have much in common with the functions $\sigma(z)$, $\tilde{\sigma}(z)$. In order to describe these general characteristics we define a general class of real functions $F \equiv F_B$ of the following description.

Let B be a domain in $2k$-dimensional $(z_1, \cdots, z_k)$-space. The class $F \equiv F_B$ is the smallest class of real functions $\psi(z)$ in B which includes all functions $\psi(z) = \log |f(z)|$ where $f(z)$ is analytic in B, and which is closed under the following operations:

(i) the operation of forming $\psi(z) = \lambda_1\psi_1(z) + \lambda_2\psi_2(z)$ for $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, and that of forming $\psi(z) = \sup_\alpha \psi_\alpha(z)$ for any set of functions $\{\psi_\alpha(z)\}$ which are collectively bounded from the right in every compact subset of B:

(ii) the operation of forming $\lim_{n\to\infty} \psi_n(z)$ for $\psi_n(z) \geq \psi_{n+1}(z)$, and

(iii) the operation of forming $\bar{\psi}(z) = \limsup_{z'\to z} \psi(z')$ for any $\psi(z)$ in F.

We now say that a function $\psi(z)$ defined in B is a *Hartogs function* in B if it belongs to $F_{B'}$ for every domain B' whose closure lies in B.

We shall prove

Theorem 7. *If $\sigma(z)$, $\tilde{\sigma}(z)$ are radii of largest or maximal Hartogs domains then* $\log \dfrac{1}{\sigma(z)}$ *and* $\log \dfrac{1}{\tilde{\sigma}(z)}$ *are Hartogs functions.*

For the proof we return to the function $f(z; w)$ of equation (27) and we introduce the functions

$$\varphi_n(z) = \frac{1}{n} \log |f_n(z)|, \qquad \varphi_0(z) = \log \frac{1}{\sigma^+(z)}, \qquad \varphi(z) = \log \frac{1}{\sigma(z)} \tag{32}$$

Obviously

$$\varphi_0(z) = \limsup_{n\to\infty} \varphi_n(z), \qquad \varphi(z) = \limsup_{z'\to z} \varphi_0(z')$$

Since (27) is continuously convergent in (28), the sequence $\{\varphi_n(z)\}$ is collectively bounded from the right in every compact subset of B. Thus $\log \dfrac{1}{\sigma(z)}$ is a Hartogs function in B. Next we introduce

$$\tilde\varphi_0(z) = \log \frac{1}{\tilde\sigma_0(z)}, \qquad \tilde\varphi(z) = \log \frac{1}{\tilde\sigma(z)}$$

We obviously have

$$\tilde\varphi_0(z) = \sup_f \varphi(z), \qquad \tilde\varphi(z) = \limsup_{z'\to z} \tilde\varphi_0(z')$$

and as before the set $\{\varphi(z)\}$ for all functions $f(z; w)$ analytic in (28) is collectively bounded from the right in every compact subset of B. Hence $\log \dfrac{1}{\tilde\sigma(z)}$ is also a Hartogs function in B. This yields Theorem 7.

The following lemmas will show how much of a restriction on $\sigma(z)$ and $\tilde\sigma(z)$ Theorem 7 implies. The nearest to it in Chapter IV was the restriction on $\tilde\rho(\zeta)$ as elucidated in Theorem 9 of that chapter and the reader will see for himself that the restriction there is much feebler than the restriction implied in Theorem 7.

Lemma 4. *If (z^0) is any point in B, and $C(z^0; \rho)$ is any polycylinder with center (z^0) and radii (ρ) whose closure lies in B, and if T is the k-dimensional point set $\zeta_j = z_j^0 + \rho_j e^{i\vartheta_j}$, $0 \le \vartheta_j < 2\pi$, then any Hartogs function $\psi(z)$ satisfies the inequality*

$$(33) \quad \psi(z^0 + z) \le \int_T P(z, \vartheta)\psi(z_j^0 + \rho_j e^{i\vartheta_j})\,d\vartheta, \qquad (z_j = r_j e^{i\varphi_j},\ r_j < \rho_j)$$

where $d\vartheta$ is $d\vartheta_1 \cdots d\vartheta_k$ and $P(z, \vartheta)$ is the Poisson kernel (14).

In the second place, if we carry out any affine transformation

$$(34) \qquad z_j = b_j + \Sigma_{p=1}^k a_{jp} z'_p, \qquad j = 1, \cdots, k$$

then (33) *also holds in the new coordinates. In other words, as viewed from the original coordinates, our inequality holds relative to any "polycylinder" in analytically oblique position.*

Finally, the inequality also holds relative to any lower dimensional polycylinder if only some of the coordinates $z_1, \cdots, z_k$ are varied and the others are held fast.

Proof. For functions of the form $\lambda \log |f(z)|$ the lemma, in its entirety, is nothing but Theorem 1, Chapter VI. Thus we have only to show that (33) is preserved, under the three closure operations for functions in $F_{B'}$. For operation (i) this is readily verified, and for

operation (ii) it is a consequence of Lemma 1. For (iii) we first note that for fixed z and any $\epsilon > 0$ there exists an $\eta > 0$ such that $|\Delta z| < \eta$ implies

$$\frac{1}{1+\epsilon} \leq \frac{P(z+\Delta z, \vartheta)}{P(z, \vartheta)} \leq 1 + \epsilon \tag{35}$$

Hence

$$\begin{aligned} \bar{\psi}(z^0 + z) &= \lim\sup_{\Delta z \to 0} \psi(z^0 + z + \Delta z) \\ &\leq (1+\epsilon) \lim\sup_{\Delta z \to 0} \int P(z, \vartheta)\psi(z^0 + \Delta z + \rho e^{i\vartheta})d\vartheta \\ &\leq (1+\epsilon)\int P(z, \vartheta)\bar{\psi}(z^0 + \rho e^{i\vartheta})d\vartheta \end{aligned}$$

and this completes the proof of the lemma.

We remark that for $k = 1$, the second and third parts of Lemma 4 add nothing to its first part. For $k = 1$, a function is called subharmonic if it satisfies relation (33) and is upper semicontinuous (as is the case with $\bar{\varphi}(z)$ and $\bar{\varphi}(z)$). For $k > 1$ several types of generalizations are available, and the one embodied in Lemma 4 is the most restrictive. For $k = 1$, if $\psi(x, y) \equiv \Psi(z, \bar{z})$ is subharmonic and has continuous partial derivatives of second order, then $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 \geq 0$, that is $\partial^2\Psi/\partial z\partial\bar{z} \geq 0$ (see section 4, Chapter II). Using this we will prove

Lemma 5. *If a Hartogs function $\psi(z_1, \bar{z}_1; z_2, \bar{z}_2, \cdots; z_k, \bar{z}_k)$ has continuous partial derivatives of second order, then*

$$\Sigma_{p=1}^{k} \Sigma_{q=1}^{k} \frac{\partial^2\psi}{\partial\bar{z}_p\partial z_q} a_p\bar{a}_q \geq 0 \tag{36}$$

for any point (z) in B, and any complex numbers $a_1, \cdots, a_k$.

We obtain (36) from $\partial^2\psi/\partial z_1\partial\bar{z}_1 \geq 0$ by way of any affine transformation (34) with $a_p \equiv a_{1p}$.

We note without proof that for differentiable functions, condition (36) is not only a necessary but also a sufficient condition for ψ to be a Hartogs function and that any Hartogs function can be obtained as a limit of such differentiable ones. However, this will be of no consequence in our contexts.

§7. Generalized Hartogs Domains

We consider a domain D in the space of complex variables $(z_1, \cdots, z_k; w_1, \cdots, w_l)$, $k \geq 1$, $l > 1$, writing it in the form

$$z \in B; \qquad w \in \Delta(z) \tag{37}$$

We now assume that all projections $\Delta(z)$ in w-space are dilations of one and the same circular domain Σ. In order to have the nearest

approach to a Hartogs domain with basis B, we assume that whenever Σ contains a point $(w_1, \cdots, w_l)$ it also contains the disc

$$(w_1 t, \cdots, w_l t) \tag{38}$$

for $|t| < 1$. If, for fixed $\sigma > 0$, we associate with each point w of Σ the disc (38) for $|t| < \sigma$ we obtain a dilation of Σ by the factor σ. Denoting the new domain by $\sigma\Sigma$, we now assume that the domain D of (37) has the form

$$D: \qquad [z \in B;\ w \in \sigma^-(z)\Sigma] \tag{39}$$

where $\sigma^-(z)$ is a suitable "Hartogs radius." If $f(z;\ w)$ is analytic in (39) then there exists a largest domain of the type

$$z \in B; \qquad w \in \sigma(z)\Sigma \tag{40}$$

into which it can be continued, and there also exists a maximal completion

$$z \in B; \qquad w \in \tilde{\sigma}(z)\Sigma$$

into which all $f(z;\ w)$ can be continued. We now claim that the functions $\log \dfrac{1}{\sigma(z)}$ and $\log \dfrac{1}{\tilde{\sigma}(z)}$ are again Hartogs functions. It will be sufficient to consider only the first function.

We will have to draw fully on the results in sections 7 and 8 of Chapter IV. We replace the variables $w_1, \cdots, w_l$ by the variables $w_1 t, \cdots, w_l t$, thus adding a new complex variable t, and this corresponds to replacing the domain (39) by a genuine Hartogs domain

$$(z,\ w) \in (B,\ \Sigma); \qquad |t| < \sigma^-(z,\ w) \tag{41}$$

where

$$\inf_{w \in \Sigma} \sigma^-(z,\ w) = \sigma^-(z)$$

The function $f(z_1, \cdots, z_k;\ w_1 t, \cdots, w_l t)$ can be continued from (41) into a largest domain

$$(z,\ w) \in (B,\ \Sigma); \qquad |t| < \sigma(z,\ w) \tag{42}$$

and it can be seen that

$$\inf_{w \in \Sigma} \sigma(z,\ w) = \sigma(z)$$

where $\sigma(z)$ is the dilation factor occurring in (40). Now by Theorem 7, the function

$$\log \frac{1}{\sigma(z,\ w)}$$

is a Hartogs function in (B, Σ), therefore, by properties of Hartogs functions, the function

$$\log \frac{1}{\sigma(z)} = \sup_{w \, \varepsilon \, \Sigma} \log \frac{1}{\sigma(z, w)} \tag{43}$$

is likewise a Hartogs function in (B, Σ). Now, the latter function is constant in the variables w, hence, by the last part of Lemma 4, this function is a Hartogs function in the domain B. This completes the proof of our assertion.

Probably the most interesting case arises if Σ is the unit sphere $|w_1|^2 + \cdots + |w_l|^2 < 1$, in which case the dilated domain $\sigma\Sigma$ is the sphere

$$|w_1|^2 + \cdots + |w_l|^2 < \sigma^2$$

§8. Half-planes instead of Circles

If in a regular Hartogs domain we replace the variable t by e^{-t}, we obtain a domain of the form

$$z \, \varepsilon \, B; \qquad \operatorname{Re}\{t\} > \rho(z) \tag{44}$$

Since the circle $|t| < \sigma$ goes into a half-plane $\operatorname{Re}\{t\} > \log \frac{1}{\sigma}$, it is reasonable to expect that, for given B, in a largest or maximal region of the type (44), the function $\rho(z)$ will be a Hartogs function. This is indeed the case. Consider a domain

$$z \, \varepsilon \, B; \operatorname{Re}\{t\} > \rho^-(z) \tag{45}$$

and a function $f(z, t)$ analytic in (45) and assume that (44) is the largest domain with basis B into which $f(z, t)$ can be continued. Consider any bounded subdomain B' of B, whose closure is contained in B. It is easy to see that for $z \, \varepsilon \, B'$, $\rho^-(z)$ is bounded from above. In order to utilize this we have formulated the definition of Hartogs function in terms of subdomains B' of B, see section 6, and therefore we may from now on assume that $\rho^-(z)$ is bounded from above in B, $\rho^-(z) < \alpha_0$. We may also assume that $\alpha_0 > 0$.

Now, for any $\alpha > \alpha_0$, we consider $f(z, t)$ in the Hartogs domain

$$z \, \varepsilon \, B; \qquad |t - \alpha| > \alpha - \alpha_0$$

which is a part of (45), and we continue it into the largest Hartogs domain

$$z \, \varepsilon \, B; \qquad |t - \alpha| < \alpha - \rho_\alpha(z)$$

The reader will verify without great difficulties that $\rho_\alpha(z)$ is monotonely

increasing with α increasing and that for the quantity $\rho(z)$ in (44) we have

$$\lim_{\alpha\to\infty} \rho_\alpha(z) = \rho(z) \tag{46}$$

The quantity $\rho_\alpha(z)$ may be $-\infty$, however if $\rho(z)$ is $> -\infty$ then $\rho_\alpha(z) > -\infty$. Now, by Theorem 7, in combination with property (i) of section 6, we conclude that the function

$$\alpha \log \frac{\alpha}{\alpha - \rho_\alpha(z)} \equiv \alpha \log \alpha + \alpha \log \frac{1}{\alpha - \rho_\alpha(z)} \tag{47}$$

is a Hartogs function for each α. In order to avoid the complication of $\rho_\alpha(z)$ being unbounded from below we introduce for each $n = 1, 2, \cdots$ the functions

$$\rho_\alpha^n(z) = \sup \left[\alpha \log \frac{\alpha}{\alpha - \rho_\alpha(z)}, \qquad \alpha \log \frac{\alpha}{\alpha + n}\right]$$

Each of them is a Hartogs function and for fixed n the set in α is bounded on the right. Hence their lim sup as $\alpha \to \infty$ is a Hartogs function. But this lim sup is merely the function

$$\rho^n(z) \equiv \sup [\rho(z), -n]$$

Obviously $\rho^n(z) \searrow \rho(z)$ as $n \to \infty$ and therefore by property (ii) of section 6 $\rho(z)$ itself is a Hartogs function.

The half-plane function $\tilde{\rho}(z)$ which describes the maximal analytic completion of the domain (45) is also a Hartogs function since $\tilde{\rho}(z)$ is obtained by the operation of taking the supremum of $\rho(z)$ for all functions $f(z, t)$ analytic in (45) followed by the smoothing operation (iii) of section 6.

§9. Borel Radii

Finally, given a fixed basis B in $(z_1, \cdots, z_k)$-space, we consider a function $f(z; t)$ in a domain of the Borel-Hartogs type

$$z \in B; \qquad |t - r^-(z)| < r^-(z) \tag{48}$$

where $r^-(z) > 0$ is any (semicontinuous) radius function. This radius and the radius $r(z)$ to come may also be $+\infty$, with the understanding that for $r = +\infty$, the circle $|t - r| < r$ shall denote the half-plane $\operatorname{Re}\{t\} > 0$.

It is trivial to see that for $r(z) \geq r^-(z)$ the domain (48) will be part of the domain

$$z \in B; |t - r(z)| < r(z) \tag{49}$$

and thus we may construct extremal domains of type (49) into which one or all functions $f(z, t)$ may be continued. We claim that for an extremal domain the function

$$\frac{1}{r(z)} \tag{50}$$

itself (although not necessarily its log) is a Hartogs function. This will follow from the following known theorem for functions in one variable.

Theorem 8. *If $f(t) = \Sigma_{\nu=0}^{\infty} a_\nu t^\nu$ is analytic in some circle $|t| \leq \delta$ and if r is the largest radius for which $f(t)$ can be continued into $|t - r| < r$ then*

$$\frac{1}{r} = \sup \left[0; \limsup_{\rho \to \infty} \frac{2}{\rho} \log |F(\rho)| \right],$$

where $F(t)$ is the Borel transform $\Sigma_{\nu=0}^{\infty} a_\nu t^\nu/\nu!$

Before inserting a proof of Theorem 8 we will draw our conclusion. If B' is a compact set in B, then there exists an $\epsilon > 0$ such that $r^-(z) > \epsilon$ for $z \in B'$. We now consider the expansions

$$f(z; \epsilon + t) = \Sigma_{\nu=0}^{\infty} \frac{f^{(\nu)}(z; \epsilon)}{\nu!} t^\nu$$

$$F_\epsilon(z; t) = \Sigma_{\nu=0}^{\infty} \frac{f^{(\nu)}(z; \epsilon)}{(\nu!)^2} t^\nu$$

and the quantities

$$[r_\epsilon^+(z)]^{-1} = \sup \left[0; \limsup_{\rho \to \infty} \frac{2}{\rho} \log |F_\epsilon(z; \rho)| \right]$$

$$r_\epsilon(z) = \inf_{z' \to z} r_\epsilon^+(z')$$

$$r(z) = \lim_{\epsilon \to 0} r_\epsilon(z)$$

the latter lim being increasing. It is not hard to verify that the function $r(z)$ thus obtained is our extremal function $r(z)$ of (49). Since $\log |F_\epsilon(z; \rho)|$ is a Hartogs function for any ϵ and ρ, it follows that (50) is also a Hartogs function.

As for the radius $\bar{r}(z)$ which describes the maximal completion of the domain (48), since $[\bar{r}(z)]^{-1}$ is the "smoothed" supremum of $[r(z)]^{-1}$ for all $f(z, t)$ in (48), it follows that it is also a Hartogs function.

Proof of Theorem 8. Putting

$$h = \limsup_{\rho \to \infty} \frac{1}{\rho} \log |F(\rho)| \tag{51}$$

our theorem will obviously follow from a combination of the following two lemmas

Lemma 6. (i) *If* $h > 0$ *then* $f(t)$ *is analytic in the circle*

$$\left|t - \frac{1}{2h}\right| < \frac{1}{2h} \tag{52}$$

(ii) *If* $h \leq 0$ *then* $f(t)$ *is analytic in the half plane* $\operatorname{Re}\{t\} > 0$.

Lemma 7. *If* $f(t)$ *is analytic in the closed circle* (53)

$$|t - r| \leq r$$

then

$$2h \leq \frac{1}{r}$$

The lemmas will be based on the well known reciprocal relations

$$F(t) = \frac{1}{2\pi i}\int_{|z|=\delta} e^{\frac{t}{\tau}} f(\tau)\,\frac{d\tau}{\tau} \tag{54}$$

and

$$f(t) = \int_0^\infty e^{-a} F(at)\,da = \frac{1}{t}\int_0^\infty e^{-\frac{b}{t}} F(b)\,db \tag{55}$$

of which the second is valid at least for t real and $0 \leq t < \delta$.

Proof of Lemma 6. By (51) if $\epsilon > 0$ there exists a positive quantity K_ϵ such that

$$|F(\rho)| < K_\epsilon e^{(h+\epsilon)\rho}; \qquad 0 < \rho < \infty$$

Hence the second integral in (55) is absolutely convergent and defines an analytic function in

$$\operatorname{Re}\left\{\frac{1}{t}\right\} > h \tag{56}$$

If $h > 0$ this means that $f(t)$ is analytic in the circle (52). If $h \leq 0$ this means that $f(t)$ is certainly analytic in the half-plane $\operatorname{Re}\{t\} > 0$. Thus Lemma 6 holds.

Proof of Lemma 7. Since f is assumed to be analytic in (a neighborhood of) the closed circle (53) there exists an $r' > r$ such that f is also analytic in the closed circle

$$|t - r| \leq r' \tag{57}$$

By making a permissible shift of the contour in (54) we may write

$$F(t) = \frac{1}{2\pi i}\int_C e^{\frac{t}{\tau}} f(\tau)\,\frac{d\tau}{\tau} \tag{58}$$

with C the circle

$$C: \qquad |t - r| = r'$$

Hence

$$F(at)e^{-a} = \frac{1}{2\pi i}\int_C e^{a\left(\frac{t}{\tau}-1\right)} f(\tau)\,\frac{d\tau}{\tau} \tag{59}$$

Now we observe that for fixed τ on C the points t for which

$$\operatorname{Re}\left\{\frac{t}{\tau}\right\} < 1 \tag{60}$$

consist of all points of the half-plane which contain the origin and which is bounded by the line through τ perpendicular to 0τ. As τ ranges over the circle C these half-planes all include the point $t = 2r$. (The lines envelope an ellipse with foci at $t = 0$, $t = 2r$.) Using this fact and the fact that $f(\tau)/\tau$ is bounded on C, and setting $t = 2r$ in (59) we conclude that

$$|F(a \cdot 2r)e^{-a}| < \text{const.} \qquad 0 \leq a < \infty$$

Recalling the definition of h (see (51)), we see that this yields

$$h \leq \frac{1}{2r}$$

Thus Lemma 7 holds.

§10. Radius of Meromorphy

Consider a function $f(z_1, \cdots, z_k, w)$ analytic in a domain

$$[z \in B;\ w \in \Delta(z)] \tag{61}$$

where $\Delta(z)$ is a circle containing the origin. For each z in B, $f(z, w)$ is analytic in $w \in \Delta(z)$. We now introduce the largest radius $M^+(z)$ having the property that in $|w| < M^+(z)$ there exists a meromorphic function of w which in the neighborhood of $w = 0$ coincides with $f(z, w)$. We will then have the following theorem.

Theorem 9. *The function* $\log \dfrac{1}{M^+(z)}$ *is a Hartogs function in* B.

Before entering on the proof we remark that a more natural quantity to introduce would be $-\log M(z)$ where

$$M(z) = \liminf_{\zeta \to z} M^+(\zeta)$$

and that one would expect the statement that in

$$[z \in B, |w| < M(z)]$$

our function $f(z, w)$ has a continuation as a meromorphic function in the $k + 1$ complex variables simultaneously. However, this statement is not as easily proved as one might expect. The complications arise from the fact that for several variables a meromorphic function is a much more elaborate concept than for $k = 1$ and we do not propose to enter into a full discussion. However, at the end of the proof we will touch upon meromorphic functions in $(k + 1)$-variables in another context. We will generalize our theorem from a function $f(z, w)$ which is analytic in (61) to one which is given as a quotient

$$f(z; w) = \frac{\varphi(z, w)}{\psi(z, w)} \tag{62}$$

of two analytic functions in (61).

Now for the proof! According to Hadamard, if $f(w) = \Sigma_{j=0}^{\infty} f_j w^j$ is analytic at $w = 0$, then its radius of meromorphy is obtained in the following way. If we introduce the determinants

$$D_j^\mu = \begin{vmatrix} f_j & f_{j+1} & \cdots & f_{j+\mu} \\ \cdots & & & \\ f_{j+\mu} & f_{j+\mu+1} & \cdots & f_{j+2\mu} \end{vmatrix} \tag{63}$$

and the numbers

$$l_\mu = \limsup_{j\to\infty} |D_j^\mu|^{\frac{1}{j}} \tag{64}$$

then the quotient

$$\frac{l_\mu}{l_{\mu-1}} \tag{65}$$

is nonincreasing with μ, and its limit is M^+, the radius of meromorphy of $f(w)$. Now in our case we have an expansion

$$f(z, w) = \Sigma_{j=0}^{\infty} f_j(z) w^j \tag{66}$$

and we introduce the corresponding quantities $D_j^\mu(z)$, $l_\mu(z)$. If S is any closed subset of B, then there exist a radius $\sigma = \sigma_S > 0$ and a (finite) constant $A = A_S$ such that (66) in absolute value is $\leq A$ in $|w| \leq \sigma$. Thus by(66)

$$|f_j(z)| \leq \frac{A}{\sigma^j}$$

and hence by a crude estimate

$$|D_j^\mu(z)| \leq \frac{(\mu + 1)!A^{\mu+1}}{\sigma^{(\mu+1)j+\mu(\mu+1)}}$$

for z in S. Hence, for fixed μ,

$$\frac{1}{j}\log |D_j^\mu(z)| \leq \log \frac{1}{\sigma^{\mu+1}} + \frac{1}{j}\log \frac{(\mu + 1)!A^{\mu+1}}{\sigma^{\mu(\mu+1)}}$$

and its limit superior as $j \to \infty$ is a Hartogs function in B. Noting that this lim sup is $\log l_\mu(z)$ we see that

$$\frac{1}{\mu}\log l_\mu(z) \leq \frac{\mu + 1}{\mu} \log \frac{1}{\sigma}$$

and thus

$$\limsup_{\mu\to\infty} \frac{1}{\mu} \log l_\mu(z)$$

is a Hartogs function in B. But since (65) is nonincreasing the limit proper exists and is equal to

$$\lim_{\mu\to\infty} \log \frac{l_\mu(z)}{l_{\mu-1}(z)} \equiv \log \frac{1}{M^+(z)}$$

This proves our theorem.

Now, in the general case (62) *we add the explicit assumption that for no z shall ψ (z, w) be identically zero in*

(67) $$w \in \Delta(z)$$

Thus for each z in B, the quotient (62) is meromorphic in w in (67), and we can introduce the largest radius of meromorphy $M^+(z)$. Now if for all z, $\varphi(z, w)$ is nonvanishing at the origin $w = 0$, the quotient is analytic, and our previous theorem applies. Suppose however that for $(z) = (0)$ say, $\varphi(0, w)$ vanishes for $w = 0$. Then by the Weierstrass preparation theorem (see section 1, Chapter IX) there exists a neighborhood

(68) $$|z \in B'; w \in \Delta'(z)]$$

of the origin such that in this neighborhood $\varphi(z, w)$ is representable in the form

$$\varphi(z, w) = [w^p + A_1(z)w^{p-1} + \cdots + A_p(z)]\Omega(z, w)$$

where $\Omega(z, w)$ is analytic and nonvanishing in (68), $A_1, \cdots, A_p$ are analytic in B', and $A_1 \cdots, A_p$ are zero at the origin. Thus we have

in (68)

$$[w^p + A_1(z)w^{p-1} + \cdots + A_p(z)]f(z, w) \equiv \frac{\varphi(z, w)}{\Omega(z, w)} \equiv g(z, w)$$

Now, for each z, the radius of meromorphy of the function $g(z, w)$, which is analytic in (68), is the same as for $f(z, w)$ itself. But to the latter our theorem applies, locally to every neighborhood B' of B, and thus our entire Theorem 9 applies to the meromorphic function as described.

§11. A Theorem on Complex Lie Groups

We have defined and treated subharmonic functions by inequalities in terms of boundary integrals. But they also obey inequalities in terms of space integrals. Thus, if we have

$$\varphi(0, 0) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(\rho \cos \theta, \rho \sin \theta) d\theta$$

for all sufficiently small ρ, then on multiplying by $2\rho d\rho$ and integrating in $0 \leq \rho \leq r$ we obtain

$$\varphi(0, 0) \leq \frac{1}{\pi r^2} \int_U \varphi(x, y) dx dy$$

where U is the circle $x^2 + y^2 \leq r^2$. In general, the analogue to this exists for Hartogs functions in several complex variables. Assume for the sake of simplicity that all functions are continuous. Then the Hartogs functions fall under the following very general definition (which is one of many possible ones). We call, for the present, a function $f(x_1, \cdots, x_n)$ in a domain D in E_n subharmonic, if corresponding to any x in D there exists a sufficiently small neighborhood U_x containing x such that

$$f(x) \leq \frac{1}{m(U_x)} \int_{U_x} f(\xi) dv_\xi$$

where $m(U_x)$ is the volume of U_x. Thus the expression on the right is the spatial average of $f(\xi)$ over U_x. For $n = 2k$, the real (an imaginary) part of any function of k complex variables is certainly subharmonic in this sense.

Now, take an orthogonal matrix

$$\{a_{pq}\}, \qquad p, q = 1, \cdots, r$$

The dimension r shall have no arithmetical connection with n. Ortho-

gonality means

(69) $$\Sigma_{s=1}^{r} a_{ps}^2 = 1, \qquad p = 1, \cdots, r$$

and

(70) $$\Sigma_{s=1}^{r} a_{ps} a_{qs} = 0, \qquad p, q = 1, \cdots, r, \qquad p \neq q$$

We have split off (69) from (70) as a condition because we will make use of (69) alone and none of (70). However in the ultimate application it would be pointless to have anything more general than orthogonality.

Now, assume that we have a family of orthogonal transformations, all coefficients $a_{pq}(x)$ being functions in the same domain D in E_n. Out main statement is as follows.

Theorem 10. *If all $a_{pq}(x)$ are subharmonic in the manner previously stated for the same neighborhoods U_x, then they are constant.*

Proof. Indeed we have

$$a_{ps}(x) \leq \frac{1}{m(U_x)} \int_{U_x} a_{ps}(\xi) dv_\xi$$

From Schwarz's inequality

(71) $$\left(\int a_{ps}(\xi) dv_\xi\right)^2 \leq m(U_x) \int [a_{ps}(\xi)]^2 dv_\xi$$

we thus obtain

(72) $$[a_{ps}(x)]^2 \leq \frac{1}{m(U_x)} \int [a_{ps}(\xi)]^2 dv_\xi$$

and hence

(73) $$\Sigma_{s=1}^{r} [a_{ps}(x)]^2 \leq \frac{1}{m(U_x)} \int \Sigma_{s=1}^{r} [a_{ps}(\xi)]^2 dv_\xi$$

Since (69) holds, (73) is a strict equality and working backwards we see that (71) must be an equality. However by the precise version of Schwarz's inequality this implies that $a_{ps}(\xi)$ is a constant in ξ as claimed.

Theorem 11. *If in a family of unitary matrices*

(74) $$\{c_{pq}(z)\}_{p,q=1,\cdots,r}$$

the coefficients are analytic functions of k complex variables in a domain D, then they are constant.

If we introduce the unitary transformation

(75) $$\zeta'_p = \Sigma c_{pq}(z) \zeta_q$$

for complex vectors ζ, ζ' and split into real and imaginary parts, then (74) gives rise to a family of orthogonal transformations in $2r$ real

variables whose coefficients are real and imaginary parts of c_{pq}. Thus Theorem 11 is a consequence of Theorem 10.

We may also obtain

Theorem 12. *If a complex Lie group is compact, or only if its adjoint group is compact, then the adjoint group is the identity and thus the group is commutative.*

For this theorem we naturally presuppose a knowledge of the prerequisites involved. We recall that the adjoint group of a complex group is a complex group of the form (75). If the group is compact, then so is the adjoint group, though the converse is not true. If the adjoint group is compact, then by a suitable choice of coordinates (ζ_q) it can be made into a subgroup of the unitary group, and Theorem 11 shows that in our case the adjoint group must be the identity. But this implies in general that the original group is Abelian.

Thus a compact complex analytic manifold which is a group space can only be the type of multi-torus as occurs in the theory of Abelian functions.

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For the theorem on complex Lie groups in section 10, see:

8. S. BOCHNER and DEANE MONTGOMERY, Groups of differentiable and real or complex analytic transformations, *Annals of Math.*, Vol. 46 (1945), pp. 685–694.

A systematic exposition of subharmonic functions (mainly in one complex variable) has been given in:

9. T. Rado, *Subharmonic functions*, Springer, 1937 (Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 5, Part 1).

It was not necessary in our context to introduce a formal definition of subharmonic functions. But in so far as such a definition is implied in our argument the reader will notice two simplifications. We do require boundedness from above but we freely admit the value $-\infty$; secondly, although we emphasize semicontinuity, we do not introduce it as a primary prerequisite. On the whole, it has been our experience that "subharmonic functions" is a fluid concept and that it gains in usefulness if adjusted to circumstances in which it arises.

CHAPTER VIII

Removable Singularities

§1. Generalized Solutions of Partial Differential Equations

In the space of n real variables take an (open) domain D and in it a closed cube

$$R: \qquad a_j \leq x_j \leq b_j, \qquad j = 1, \cdots, n$$

Take a continuous differentiable function $f(x)$ in D which vanishes in $D - R$. Then for each j, we have for the volume integral the relation

$$\int_D \frac{\partial f}{\partial x_j}\, dv_x \equiv 0 \tag{1}$$

In fact, since $\dfrac{\partial f}{\partial x_j} \equiv 0$ in $D - R$ we have

$$\int_D \frac{\partial f}{\partial x_j}\, dv_x \equiv \int_R \frac{\partial f}{\partial x_j}\, dv_x$$

and taking the integral as a repeated integral and integrating first with respect to x_j we obtain (1). Now take two continuous, absolutely integrable functions $F(x)$, $G(x)$ in D, and assume that there exist functions $f_1, \cdots, f_n$ in D, each vanishing in $D - R$, such that identically in D we have

$$F(x) - G(x) \equiv \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \cdots + \frac{\partial f_n}{\partial x_n} \tag{2}$$

Then we conclude

$$\int_D F(x)\, dv_x = \int_D G(x)\, dv_x \tag{3}$$

We will apply this in the following manner. Consider an operator

$$\Lambda f \equiv \Sigma_{r_1+\cdots+r_n \leq N} a_{r_1 \cdots r_n}(x) \frac{\partial^{r_1+\cdots+r_n} f}{\partial x_1^{r_1} \cdots x_n^{r_n}} \tag{4}$$

that is, a *finite* sum of the form

$$af + b_1 \frac{\partial f}{\partial x_1} + \cdots + b_n \frac{\partial f}{\partial x_n} + c\frac{\partial^2 f}{\partial x_1^2} + \cdots \tag{5}$$

This is the most general *linear* differential operator, the coefficients $a_{r_1 \cdots r_n}(x)$ being assumed differentiable as often as required for any one purpose. In our essential applications these coefficients will be constants, but some of our results, and indeed the principal result will apply to nonconstant coefficients. Therefore for the sake of completeness we take nonconstant coefficients. The index N need not always be the smallest one possible. Thus it may sometimes happen that

$$a_{r_1 \cdots r_n}(x) \equiv 0$$

for all combinations $r_1, \cdots, r_n$ with $r_1 + \cdots + r_n = N$. But usually it will not, and we will always call N the order of the operator (4). Furthermore, both the coefficients $a_r(x)$ and all functions $f(x)$ might be complex-valued. But usually they will be real valued. In either case they will not be presumed analytic in the real variables x unless so stated.

There always exists an *adjoint* operator, which will be denoted by Λ^*, which we will let operate on a function which will be designated by another letter, say φ. The formal definition is

$$\Lambda^*\varphi = \Sigma_{r_1+\cdots+r_n \le N}(-1)^{r_1+\cdots+r_n} \frac{\partial^{r_1+\cdots+r_n}(a_r\varphi)}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} \tag{6}$$

The relation between Λf and $\Lambda^*\varphi$ is as follows. There exists a finite number of expressions

$$P_k\left(a, \frac{\partial a}{\partial x}, \cdots, f, \frac{\partial f}{\partial x}, \cdots, \varphi, \frac{\partial \varphi}{\partial x}, \cdots\right) \tag{7}$$

each being a polynomial with constant coefficients in the quantities a, f, φ and their partial derivatives up to a certain (finite) order, such that

$$(\Lambda f)\varphi - (\Lambda^*\varphi)f \equiv \frac{\partial P_1}{\partial x_1} + \cdots + \frac{\partial P_n}{\partial x_n} \tag{8}$$

this being an identity in a, f, φ and their partial derivatives (as far as the latter occur). Furthermore, in each of the polynomials (7) each individual monomial actually contains f or a derivative, and similarly φ or a derivative.

Now, assume that f and φ are differentiable up to order N, and that the function φ (and therefore its derivatives) vanishes outside

the cube R. Then (8) is a relation of the form (2), and from (3) we obtain

$$\int_D(\Lambda f) \cdot \varphi dv_x \equiv \int_D f \cdot (\Lambda^* \varphi) dv_x \tag{9}$$

The functions f and φ will play very distinct roles. The function f will be a fixed function in D, at first of differentiability class C^N. However φ will be a variable function, of the sort which we will call a *testing function*. Given a point P in D and any sufficiently small neighborhood U of P, then a testing function shall be continuous and infinitely differentiable in all of D, equal to 0 outside U, but $\neq 0$, say $= 1$, at P. Such functions have been constructed in section 3, Chapter VI. Their role is to "test" whether a given function $g(x)$ in D is everywhere 0. Obviously, if $g(x)$ is continuous, and if for all testing functions φ the relation

$$\int_D g \varphi dv_x = 0$$

holds, then g is zero. Otherwise there would be a neighborhood U in which $g(x) \geq \epsilon_0 > 0$ (or $g(x) \leq -\epsilon_0 < 0$) and for a suitable testing function φ we would have $\int_D g \varphi dv_x > 0$.

Now, in particular, a function f is a solution of the equation

$$\Lambda f = 0 \tag{10}$$

if and only if for all testing functions φ we have

$$\int_D (\Lambda f) \varphi dv_x = 0 \tag{11}$$

By (9) this is entirely equivalent with

$$\int_D f \cdot (\Lambda^* \varphi) dv_x = 0 \tag{12}$$

The difference between (11) and (12) is thus that in (12) only f itself, but none of its derivatives, is being "tested." However, the equivalence between (11) and (12) does require that f be of class C^N.

This suggests, however, a generalization of the concept of a solution of (10). We will not push the generalization to the extreme limit possible. We will require once for all that $f(x)$ shall be defined and measurable in D except for a set of measure zero, and that in every closed subcube R, $f(x)$ shall be Lebesgue integrable. We do not require that $f(x)$ have a finite Lebesgue integral $\int |f(x)| dv_x$ in all of D. On the other hand we do require that $f(x)$ shall be a Lebesgue-measurable point function, although we could replace $f(x)$ by a distribution, that is a set function $F(A)$. With this in mind we then define $f(x)$ to be a *generalized solution of* (10), if corresponding to every point P in

D there exists a neighborhood U_0 of P such that for a subneighborhood U of U_0 and for any testing function $\varphi(x)$ vanishing outside of U we shall have relation (12); the integral (12) obviously existing. Clearly if a generalized solution happens to belong to class C^N, it is a "strict" solution.

§2. Constant Coefficients

In the case of constant coefficients $a_{r_1 \cdots r_n}$, every generalized solution is a certain limit of strict solutions. If the approximating sequence of strict solutions is $f_1(x), f_2(x), \cdots$, then in every cube R we have

$$\lim_{s\to\infty} \int_R |f(x) - f_s(x)| dv_x = 0 \tag{13}$$

Actually it will suffice to prove less. We will prove (13) in every cube R, however we will allow the sequence $\{f_s\}$ possibly to differ from cube to cube, and we thus will claim the existence of the functions $\{f_s(x)\}$ and the occurrence of the relations

$$\Lambda f_s(x) = 0 \tag{14}$$

only for any one cube R.

Take a testing function $\varphi(x)$. It vanishes outside some U. Denote the distance between the boundaries of U and D by $\delta > 0$. Then for any "vector" t whose length does not exceed $\delta/2$ the translated function $\varphi_t(x) = \varphi(x - t)$ is likewise a testing function, albeit in $U_t = U + (t)$. Furthermore, *due to the constancy of the coefficients* we have

$$\Lambda^*\varphi_t(x) = (\Lambda^*\varphi(\xi))_{\xi=x-t}$$

Thus we obtain from (12)

$$\int_D f(x) \cdot (\Lambda^*\varphi_t(x)) dv_x = 0 \tag{15}$$

for $||t|| \leq \delta/2$. It is now not hard to see that we can shift the translation from φ onto f, thus obtaining

$$\int_D f(x + t)\Lambda^*\varphi(x) dv_x = 0 \tag{16}$$

for $||t|| \leq \delta/2$. Now if D_ρ denotes any subdomain of D whose boundary has a distance $\geq \rho$ from the boundary of D itself, then we can form the function

$$f_h(x) = \frac{1}{(2h)^n} \int_{-h}^{h} \cdots \int_{-h}^{h} f(x_1 + t_1, \cdots, x_n + t_n) dt_1 \cdots dt_n \tag{17}$$

for x in D_{3h}. Relation (16) implies

$$\int_{D_{2h}} f(x+t) \cdot \Lambda^*\varphi(x)dv_x = 0 \tag{18}$$

for $\varphi(x)$ a testing function in D_{2h}, and if we integrate (18) with respect to t and interchange integrations with respect to t and x, we obtain

$$\int_{D_{2h}} f_h(x) \cdot \Lambda^*\varphi(x)dv_x = 0 \tag{19}$$

Thus in D_{2h}, $f_h(x)$ is again a generalized solution of (10). But, for fixed h, $f_h(x)$ is a continuous function in x, as can be found by verifying directly the definition of continuity. Furthermore, for each cube R in D_{2h_0}, h_0 fixed, and for any sequence $h_s \to 0$ we obtain

$$\int_R |f(x) - f_{h_s}(x)| dv_x \to 0 \tag{20}$$

In fact from

$$|f(x) - f_h(x)| \leq \frac{1}{(2h)^n} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x) - f(x+t)| dv_t$$

we conclude by Fubini's theorem that

$$\int_R |f(x) - f_h(x)| dv_x \leq \frac{1}{(2h)^n} \int_{-h}^{h} \cdots \int_{-h}^{h} \psi(t_1, \cdots, t_n) dv_t \tag{21}$$

where

$$\psi(t) = \int_R |f(x) - f(x+t)| dv_x$$

But $\psi(t)$ tends to zero as $t \to 0$ by a fundamental theorem of the Lebesgue theory, and therefore (21) implies (20).

Now, starting from the generalized continuous solutions $f_s(x)$ we can iterate the process of averaging thus defining

$$f_{h,2}(x) = \frac{1}{(2h)^n} \int_{-h}^{h} \cdots \int_{-h}^{h} f_h(x_1 + t_1, \cdots, x_n + t_n) dv_t$$

These functions are of class C^1, and similarly the $(N+1)$-th h-average, $f_{h,N+1}(x)$, gives functions of class C^N which are generalized solutions of (10) and thus strict solutions. And there are sequences $\{f_s\}$ among them for which (13) holds.

§3. Harmonic Functions

There are equations (10) which are such that their strict solutions are automatically analytic, and such that if a sequence of their strict solutions converges in norm in the sense of equation (13), then the limit function, after suitable correction on a null set, is differentiable sufficiently often, and is thus itself a strict solution. The prototype

is the Laplace equation

$$\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0 \tag{22}$$

Let $f(x)$ be a strict solution of (22), that is, a harmonic function. At each point x such a function is the average over the $(n-1)$-dimensional boundary of any sphere with center at x. We write it in the form

$$f(x) = \int f(x + \rho\xi)\, d\omega_\xi$$

where ξ is a point of the unit sphere, $d\omega_\xi$ is the $(n-1)$-dimensional area-element divided by the total area, and ρ is an arbitrary small positive number. If we multiply by $\rho^{n-1}d\rho$ and integrate from 0 to r we obtain

$$f(x) = \frac{1}{\omega_n} \int_{S_r} f(x + t)\, dv_t$$

where the integration is now over the solid interior of the sphere S_r: $t_1^2 + \cdots + t_n^2 < r^2$, and ω_n is the volume of S_r. Now for a sequence of harmonic functions we obtain

$$f_j(x) = \frac{1}{\omega_n} \int_{S_r} f_j(x + t)\, dv_t, \qquad j = 1, 2, 3, \cdots \tag{23}$$

and if (13) holds and r is fixed, then (23) converges towards

$$\frac{1}{\omega_n} \int_{S_r} f(x + t)\, dv_t \tag{24}$$

Also it is not hard to see that for a fixed sufficiently small radius r, the convergence of the right side of (23) is uniform in every sufficiently small neighborhood. From this we conclude that the sequence $f_j(x)$ converges at every point x and uniformly in every neighborhood. However, as for $n = 2$, so also for $n \geq 3$, if a sequence of harmonic functions converges uniformly then it is a compact analytic class and the limit is again analytic. This follows from Poisson's integral which for the interior of the unit-sphere around the origin is

$$f(x) = C_n \int_S \frac{1 - r^2}{|x - \xi|^{n-2}} f(\xi)\, d\omega_\xi \tag{25}$$

where S is the boundary of the unit sphere, $|x - \xi|$ is the Euclidean distance of the points x, ξ; r^2 is $|\xi - 0|^2$, $d\omega_\xi$ is the area-element, and

C_n is a numerical constant. For a very elegant derivation of (25), the reader is referred to a paper of C. Carathéodory listed as number 3 at the end of this chapter.

Summarizing, we have seen for the case of constant coefficients that every generalized solution is a certain limit of strict solutions, and for the harmonic case this limit is again harmonic. Thus *every generalized solution of* (22) *is also a strict solution.*

§4. SYSTEMS OF EQUATIONS AND OF FUNCTIONS

Take a system of operators

$$\Lambda_1 f, \quad \cdots, \quad \Lambda_r f \tag{26}$$

each one of which is an expression of the type (4), and consider the system of equations

$$\Lambda_\rho f = 0, \qquad \rho = 1, \cdots, r \tag{27}$$

Obviously any solution is also a solution of the system

$$\int f \cdot (\Lambda_\rho^* \varphi) dv = 0 \tag{28}$$

and a generalized solution of the system (27) can again be defined by means of the system (28).

But now take not one unknown function $f(x)$ but a system of s such functions which we can interpret as a vector

$$f(x) = (f_1(x), \cdots, f_s(x)) \tag{29}$$

Consider a rectangular matrix of operators

$$\Lambda_{\rho\sigma}, \qquad \rho = 1, \cdots, r; \qquad \sigma = 1, \cdots, s \tag{30}$$

and consider the system of r equations

$$\Lambda_{\rho 1} f_1 + \cdots + \Lambda_{\rho s} f_s = 0 \tag{31}$$

each of them being an equation in the same s functions. If we take a test function $\varphi(x)$, which is a single function, and *not* a vector function, then our reasoning shows that a vector $f(x)$, which is sufficiently often differentiable, is a common solution of (31) if and only if we have the system of r equations

$$\int_D (f_1 \cdot \Lambda_{\rho 1}^* \varphi + f_2 \cdot \Lambda_{\rho 2}^* \varphi + \cdots + f_s \cdot \Lambda_{\rho s}^* \varphi) dv_x = 0 \tag{32}$$

Again, a generalized solution is one in which each component f is Lebesgue measurable and integrable in every neighborhood. Con-

sider, for instance, in 3-space a joint solution of the system

$$\operatorname{div} f = 0, \qquad \operatorname{rot} f = 0 \tag{33}$$

that is of the four equations

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = 0; \qquad \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 0; \qquad 1 \leq i < j \leq 3 \tag{34}$$

A generalized solution is one for which

$$\int_D \left(f_i \frac{\partial \varphi}{\partial x_j} - f_j \frac{\partial \varphi}{\partial x_i} \right) dv = 0 \tag{35}$$

$$\int_D \left(f_1 \frac{\partial \varphi}{\partial x_1} + f_2 \frac{\partial \varphi}{\partial x_2} + f_3 \frac{\partial \varphi}{\partial x_3} \right) dv = 0$$

Now, for a strict solution of (33), the known relation

$$\nabla^2 f = \operatorname{grad} \operatorname{div} f - \operatorname{rot} \operatorname{rot} f \tag{36}$$

implies the equations

$$\Delta f_1 = 0, \qquad \Delta f_2 = 0, \qquad \Delta f_3 = 0 \tag{37}$$

and thus in particular every joint solution of (33) is analytic. Now, it is an important fact, which we will prove immediately, that any generalized solution of (33) is also a generalized solution of (36) and thus of (37). This in turn means that it is analytic and hence a strict solution of (33).

Consider an operator Λ and another operator which we will denote by L, and apply them successively, thus forming $L[\Lambda f]$. On applying (9) first to L and then to Λ we obtain

$$\int L[\Lambda f] \cdot \varphi dv = \int \Lambda f \cdot L^* \varphi \cdot dv = \int f \cdot \Lambda^* L^* \varphi \cdot dv$$

and thus $(L\Lambda)^* = \Lambda^* L^*$. Suppose now that we are given a matrix of operators (30), and a system of operators $L_1, \cdots, L_r$, and we form the operator

$$Mf \equiv \Sigma_{\sigma=1}^{s} (\Sigma_{\rho=1}^{r} L_\rho \Lambda_{\rho\sigma}) f_\sigma \tag{38}$$

then any strict solution of (31) which has sufficiently many derivatives is also a solution of

$$Mf \equiv 0 \tag{39}$$

We call the system (39) an induced system, induced by the system (31) that is. We claim that a generalized solution of (31) is also a general-

ized solution of (39). In fact

$$\int \Sigma_{\sigma=1}^{s} f_\sigma (\Sigma_{\rho=1}^{r} \Lambda_{\rho\sigma}^{*} L_\rho^{*} \varphi_\rho) dv = \Sigma_{\rho=1}^{r} (\int \Sigma_{\sigma=1}^{s} f_\sigma \Lambda_{\rho\sigma}^{*} \varphi_\rho^{*}) dv$$

where $\varphi_\rho^* = L_\rho^* \varphi_\rho$, and this vanishes because of (32) if we replace there φ by φ_ρ^* and sum on ρ.

We may draw from this the previously mentioned conclusion that every generalized solution of (33) is a strict solution, consisting of harmonic functions. However of greater importance to us will be the following theorem.

Theorem 1. *In the space of 2k real variables x_j, y_j; $z_j = x_j + iy_j$, $j = 1, \cdots, k$, every generalized pair of solutions (u, v) of the system of Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial y_j} = 0, \qquad \frac{\partial u}{\partial y_j} + \frac{\partial v}{\partial x_j} = 0 \tag{40}$$

in any domain D is, after alterations on a point-set of 2k-dimensional Lebesgue measure zero, a strict solution, and thus $f(x, y) = u + iv$ is an analytic function of the complex variables $z_1, \cdots, z_k$.

This theorem is of the Menchoff-Looman type, but not at all as deep as that theorem. Nevertheless it will admit important applications.

Other theorems of a similar type are given in the paper listed as number 1 at the end of this chapter.

§5. NULLIFIERS

A nullifier will be an auxiliary function which will excise an alleged singularity of an analytic function and then in the process of the excision demonstrate that the threatening singularity was no singularity after all, in other words that it was only a removable singularity.

In E_n we take a fixed function $\varphi(x_1, \cdots, x_n)$ having the following properties:

(i) It is defined and it has (mixed and iterated) partial derivatives of every order in the entire space, and (ii) it has the value 0 in $x_1^2 + \cdots + x_n^2 \leq 1$, the value 1 for $x_1^2 + \cdots + x_n^2 \geq 2$, and values between 0 and 1 in the shell

$$1 \leq x_1^2 + \cdots + x_n^2 \leq 2$$

It is easy to construct such an even function for $n = 1$, that is, on the straight line. Having done this we can then obtain one for arbitrary n for instance by replacing in the one-dimensional function the variable x by $+\sqrt{x_1^2 + \cdots + x_n^2}$. To each $N = 0, 1, 2, \cdots,$

there obviously exists a constant C_N such that for

(41) $$j_1 + \cdots + j_n \leq N$$

we have

(42) $$\left|\frac{\partial^{j_1 + \cdots + j_n}\varphi}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}\right| \leq C_N$$

in the entire space E_n.

Next, take a finite number of points

(43) $$P_\rho = \{\xi_{\rho 1}, \cdots, \xi_{\rho n}\}, \qquad \rho = 1, \cdots, r$$

a finite number of weights $\mu_1, \cdots, \mu_r$, ($\mu_\rho \geq 0$, $\mu_1 + \cdots + \mu_r > 0$), and an $\epsilon > 0$, and form the function

(44) $$\varphi_\epsilon(x) = \frac{1}{\mu_1 + \cdots + \mu_r} \Sigma_{\rho=1}^r \mu_\rho \varphi\left(\frac{x_j - \xi_{\rho j}}{\epsilon}\right)$$

Also denote by A_η the η-neighborhood of the sets

$$A = (P_1, \cdots, P_r)$$

Thus A_η is the point set-union of the r open spheres of radius η with centers at P_ρ, $\rho = 1, \cdots, r$. We now claim that $\varphi_\epsilon(x)$ has the following properties which the reader will very easily verify:

(i) $$0 \leq \varphi_\epsilon(x) \leq 1 \text{ in } E_n$$

(ii) $$\varphi_\epsilon(x) = 0 \text{ in } A_\epsilon$$

(iii) $$\varphi_\epsilon(x) = 1 \text{ outside } A_{2\epsilon}$$

(iv) $$\left|\frac{\partial^{j_1 + \cdots + j_n}\varphi_\epsilon(x)}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}\right| \leq \frac{C}{\epsilon^{j_1 + \cdots + j_n}} \qquad \text{for} \qquad j_1 + \cdots + j_n \leq N$$

where C is the same constant as in (42).

Now take an arbitrary closed point set A, and introduce its neighborhoods A_ϵ and $A_{2\epsilon}$ for fixed ϵ. Corresponding to any η there exists a finite pointset $A^{(\eta)}$ which is a subset of A, and such that each point of A has a distance $\leq n$ from some point of $A^{(\eta)}$. For given N construct a function $\varphi_\epsilon^{(\eta)}(x)$ as before, pertaining to $N + 1$ instead of N. For a decreasing sequence $\eta_p \to 0$, the functions $\varphi_\epsilon^{(\eta_p)}$ and their derivatives of order $\leq N$ are an equiuniform class. Also their neighborhoods $A_\epsilon^{(\eta_p)}$, $A_{2\epsilon}^{(\eta_p)}$, for fixed ϵ, converge toward A_ϵ, $A_{2\epsilon}$ in a well defined sense, and we thus obtain a function φ_ϵ having our previous four properties for any bounded closed set A. These functions will be our *nullifiers*.

§6. Removable Singularities in General

Let D be a bounded open set in E_n, and let D_0 be a relatively open subset; the difference $D - D_0$ will be denoted by A. The set A is bounded and even though it is only relatively closed, our previous construction of nullifiers goes through for A. We will look upon A as an "exceptional set." In our actual application it will have to be a "small" set, of lower dimension $< n$. But at first we do not qualify it at all. It may very well have an interior.

Now take a system of equations (31) in all of D, that is, the coefficients $a_r(x)$ shall be given in all of D. Take, however, a *generalized* solution of the system in D_0. It shall be so given only in D_0, thus in particular each component shall be defined in D_0 except for a null set, and (32) shall hold (over D_0) for each testing function in D_0. We now define the functions $f_1, \cdots, f_s$ to be functions in all of D; this we do merely by putting them equal to zero in all of A. If A has Lebesgue measure zero, this process of extension is redundant. But now we ask the following question. Under what conditions is the (extended) system a generalized solution of (31) in all of D, in other words, under what conditions is (32) satisfied for testing functions in all of D? If the latter situation arises we say that the point set A consisted of removable singularities for the original functions in D_0 (or D, if A had measure zero).

If φ is a testing function in D then

$$\varphi(x)\varphi_\epsilon(x)$$

is a testing function in D_0, since it vanishes outside a subneighborhood of D_0, due to property (ii) of $\varphi_\epsilon(x)$. Since f was assumed to be a generalized solution of (31) in D_0, then for each fixed ρ we have

$$\int_D (f_1 \cdot \Lambda^*_{\rho 1}(\varphi\varphi_\epsilon) + \cdots + f_s \cdot \Lambda^*_{\rho s}(\varphi\varphi_\epsilon))dv = 0$$

If we take into consideration that the partial derivatives of φ of order $\leq N$ are bounded in D, and if we make use of all properties of $\varphi_\epsilon(x)$, it is not hard to see that for each x in D we have

$$|\Lambda^*_{\rho\sigma}(\varphi\varphi_\epsilon) - \Lambda^*_{\rho\sigma}(\varphi)| \leq \frac{C}{\epsilon^N} \tag{45}$$

However for x in $D - A_{2\epsilon}$ we even have

$$|\Lambda^*_{\rho\sigma}(\varphi\varphi_\epsilon) - \Lambda^*_{\rho\sigma}(\varphi)| \equiv 0$$

since $\varphi_\epsilon(x) \equiv 1$ for x outside $A_{2\epsilon}$. Therefore

$$\begin{aligned}|\textstyle\int_D(f_1 \cdot \Lambda^*_{\rho_1}\varphi + \cdots + f_s\Lambda^*_{\rho r}\varphi)dv| \\ &\leq \textstyle\int_D |\Sigma_\sigma f_\sigma(\Lambda^*_{\rho\sigma}\varphi - \Lambda_{\rho\sigma}(\varphi\varphi_\epsilon)| dv \\ &\leq \frac{C}{\epsilon^N} \textstyle\int_{A_{2\epsilon}}(|f_1| + \cdots + |f_s|)dv\end{aligned}$$

But if we denote by $B_{2\epsilon}$ the part of $A_{2\epsilon}$ which lies in D_0, remembering that $f_\sigma = 0$ in $D - D_0$, and if we replace 2ϵ by ϵ we finally obtain the following important theorem.

Theorem 2. *If a vector function f is a generalized solution of a system of order N in a bounded open set $D_0 = D - A$ where D is a larger open set and A is an exceptional set in it; if B_ϵ is the ϵ-neighborhood of A in D_0, if we denote by $v(\epsilon)$ the n-dimensional volume (Lebesgue measure) of B_ϵ and if we denote by $T(\epsilon)$ the strong average of f over B_ϵ, that is,*

$$T(\epsilon) = \frac{1}{v(\epsilon)} \int_{B_\epsilon}(|f_1| + \cdots + |f_s|)dv \tag{46}$$

then adding the values $f \equiv 0$ on A will produce a generalized solution of our system in all of D provided we have

$$T(\epsilon) = o\left(\frac{\epsilon^N}{v(\epsilon)}\right) \tag{47}$$

§7. Removable Singularities of Analytic Functions

Of greatest interest is the case in which the system of equations (31) is such that every generalized solution is automatically a strict solution, and analytic say. In this case, whenever A has measure zero and Theorem 2 applies, the given vector f in D_0 will automatically determine additional values on A (other than zero). And in this case our theorem is truly a theorem on removable singularities in the literal sense.

Now put $n = 2k$, and take a function $f(z_1, \cdots, z_k)$ which is analytic in D_0. When can it be continued into D? The answer given by our theorem is obtained as follows. If $f = u + iv$, then u and v satisfy the system (40) and thus $N = 1$. If now for instance $f(z)$ is bounded in D, then $T(\epsilon)$ is bounded and thus (47) will be satisfied whenever

$$1 = o\left(\frac{\epsilon}{v(\epsilon)}\right)$$

that is, whenever

$$v(\epsilon) = o(\epsilon) \tag{48}$$

or written more explicitly

$$v(B_\epsilon) = o(\epsilon) \tag{49}$$

We will later have *precise* conditions for the validity of (49) or at least for the conclusion desired. At present we only state heuristically that (49) will not be valid if the exceptional set A has the highest dimension compatible with being a hypersurface, namely $n - 1$. Because in this case, B_ϵ is an n-dimensional layer of thickness 2ϵ and thus $v(\epsilon)$ is proportional to ϵ which is not compatible with (48). However, any slight additional bit suffices. Thus if we assume that A is only $(n - 2)$-dimensional, then $v(\epsilon)$ is proportional to ϵ^2 and (48) is amply verified. For instance in the case of one complex variable $n = 2$, a zero-dimensional set A consists of one or several exceptional points, and for a bounded analytic function, they are accordingly removable singularities. This then is the classical theorem of Riemann. But the customary proof is one employing Cauchy's integral, that is curve integrals, whereas we draw our conclusion from what corresponded to a solid integral. Returning to (47), if A is $(n - 1)$-dimensional, then our conclusion still holds if $f(x)$ converges to zero as x approaches A, instead of only remaining bounded.

If, however, we consider not analytic functions of complex variables, but say harmonic functions, that is solutions of (22), then $N = 2$; and in this case our conclusions "worsen" by one dimensional unit. Thus even if A is a $(n - 2)$-dimensional smooth surface, then mere boundedness is not enough to ensure a removal of the singularity. And for $(n - 1)$-dimensional A it is not enough to know that $f(x)$ has the value zero everywhere on A. Thus for $n = 2$ the harmonic function $f(x, y) \equiv |x|$ is harmonic for $x < 0$ and for $x > 0$ with $f(0, y) \equiv 0$ and yet it is not harmonic throughout. Furthermore, in three-space a harmonic function might have proper singularities along a curve if it stays bounded but not if it has boundary values zero on the curve.

Before continuing we will derive a very general uniqueness theorem.

Theorem 3. *Let D_0 be a bounded domain part of whose boundary is formed by a piece B of an $(n - 1)$-dimensional surface, and denote by B_ϵ the ϵ-neighborhood of B in D_0. If a system of equations of order N is such that every generalized solution is automatically analytic, then*

whenever $f(x)$ has "boundary value zero" on B in the precise sense that

$$T(\epsilon) = o\left(\frac{\epsilon^N}{v(\epsilon)}\right) \tag{50}$$

then $f(x) \equiv 0$ throughout.

It should be noted that for B sufficiently smooth, $v(\epsilon) \sim C\epsilon$, and (50) means

$$T(\epsilon) = o(\epsilon^{N-1})$$

which for instance for $N = 1$ will be satisfied if $f(x)$ has boundary values zero on B. If however $N > 1$ we will obtain this if we assume that on approaching the boundary points B we have $f(x) = o(\delta^{N-1})$, where δ is the distance of x from B.

For the proof of Theorem 3 take a domain D_1 which borders on B from the outside, and then put $D = D_0 + B + D_1$, and $A = B + D_1$, and complete $f(x)$ by zero in A. We can now apply Theorem 2, and we obtain the conclusion that the vector function $f(x)$ thus completed is a strict solution of the system in all of D. In particular it is analytic in D, but being zero in a subdomain D_1, it vanishes identically.

§8. A Type of Exceptional Set

Let $\varphi(x_1, \cdots, x_n)$ be a function in a domain D of E_n. The function is said to satisfy a *uniform Lipschitz condition of order* λ, $0 \leq \lambda \leq 1$, in D if there exists a constant M such that

$$\left|\varphi(x_1', \cdots, x_n') - \varphi(x_1, \cdots, x_n)\right| < M \cdot \max_{j=1,\ldots,n} \left|x_j - x_j'\right|^\lambda$$

for every pair of points x', x in D. It is clear that the condition implies uniform continuity of $\varphi(x)$ in D. Also by the mean value theorem for several real variables, it is clear that if $\varphi \epsilon K^{(1)}$ in some convex domain T containing the closure of D, then φ satisfies a uniform Lipschitz condition of order 1 in D, the bound M being any bound for the first derivative of $\varphi(x)$ *in all directions*, for an open neighborhood of D in T.

Theorem 4. *Let*

$$\varphi_1(x_1, \cdots, x_n), \qquad \cdots, \qquad \varphi_{n+2}(x_1, \cdots, x_n)$$

be $(n + 2)$ real-valued functions in a domain D of E_n, each satisfying a uniform Lipschitz condition of order $\lambda (0 \leq \lambda \leq 1)$. Let $(a_1, \cdots, a_n)$ be a point in D and, for ρ sufficiently small, consider the cube

$$R_\rho: \qquad -\rho \leq x_j - a_j \leq \rho, \qquad j = 1, \cdots, n$$

Then the set A of points (y) in the $(n + 2)$-dimensional space $(y_1, \cdots, y_{n+2})$ defined by

$$A: \quad y_\alpha = \varphi_\alpha(x_1, \cdots, x_n), \ (x) \text{ in } R_\rho; \qquad \alpha = 1, \cdots, n + 2 \tag{51}$$

has the property that

$$v(A_\epsilon) = O(\epsilon^{n+2-\frac{n}{\lambda}})$$

where A_ϵ is the ϵ-neighborhood of A in E_{n+2}, and $v(A_\epsilon)$ is its Lebesgue measure. In particular for $\lambda > n/(n + 1)$, and thus certainly for $\lambda = 1$, we have

$$v(A_\epsilon) = o(\epsilon) \tag{52}$$

For the proof we may assume $a_1 = \cdots = a_n = 0$. Let

$$\epsilon_0 = \frac{1}{\sqrt{n}} \operatorname{dist} [R_\rho, \text{boundary of } D] \tag{53}$$

and let ϵ be a fixed positive number $\leq \epsilon_0$. With this ϵ we define a system of points in D each of which will be the center of a cube of side ϵ, and the sum of these cubes will cover R_ρ and be contained in D. These lattice points are merely the points

$$(r_1\epsilon, \cdots, r_n\epsilon) \tag{54}$$

where $r_1, \cdots, r_n$ range independently over $0, \pm 1, \pm 2, \cdots, \pm[\rho/\epsilon]$, where $[\rho/\epsilon]$ denotes the largest integer $\leq \rho/\epsilon + \frac{1}{2}$. This yields

$$N_\epsilon = \left(2\left[\frac{\rho}{\epsilon}\right] + 1\right)^n$$

lattice points. We arrange them in any order and denote them by

$$(x_1^p, \cdots, x_n^p), \qquad p = 1, \cdots, N_\epsilon$$

With each of these lattice points as center we define a cube of side ϵ,

$$R^{(p)}: \qquad |x_j - x_j^p| \leq \frac{\epsilon}{2}, \qquad j = 1, \cdots, n \tag{55}$$

The cubes clearly cover R_ρ, and moreover by (53), since $\epsilon \leq \epsilon_0$, they all lie within D. Now let $A^{(p)}$ denote the map of $R^{(p)}$ by means of the transformation

$$y_\alpha = \varphi_\alpha(x_1, \cdots, x_n), \qquad \alpha = 1, \cdots, n + 2 \tag{56}$$

Since the N_ϵ cubes $R^{(p)}$ cover R_ρ, it follows that these N_ϵ sets $A^{(p)}$ cover the set A defined by (51).

Now since the φ's satisfy uniform Lipschitz conditions of order λ in D it follows that there is a constant M independent of $\epsilon(\epsilon < \epsilon_0)$ such that

$$(57) \quad |\varphi_\alpha(x_1, \cdots, x_n) - \varphi_\alpha(x_1^p, \cdots, x_n^p)| < M\epsilon^\lambda, \qquad \alpha = 1, \cdots, n+2$$

for x in the cube $R^{(p)}$ defined in (55). Thus the set $A^{(p)}$ is certainly contained in the set

$$(58) \qquad |y_\alpha - y_\alpha^p| < M\epsilon^\lambda, \qquad \alpha = 1, \cdots, n+2$$

where y^p is the map of x^p by (56). If we enlarge each of the sides of the cubes (58) by an amount ϵ^λ then not only A but also A_{ϵ^λ} will be contained in the sum of the sets so obtained, that is A_{ϵ^λ} is contained in the sum of the cubes

$$\tilde{A}^{(p)}: \qquad |y_\alpha - y_\alpha^p| < (M+1)\epsilon^\lambda, \qquad \alpha = 1, \cdots, n+2$$

It now remains to evaluate the measure of the sum of the N_ϵ cubes $\tilde{A}^{(p)}$. Each such cube has the volume

$$(M+1)^{n+2}\epsilon^{\lambda(n+2)}$$

and hence

$$\begin{aligned} v(A_{\epsilon^\lambda}) &\leq N_\epsilon \cdot (M+1)^{n+2}\epsilon^{\lambda(n+2)} \\ &= \left(2\left[\frac{\rho}{\epsilon}\right]+1\right)^n (M+1)^{n+2}\epsilon^{\lambda(n+2)} = O(\epsilon^{\lambda(n+2)-n}) \end{aligned}$$

Thus

$$v(A_\epsilon) = O(\epsilon^{n+2-\frac{n}{\lambda}})$$

This concludes the proof of Theorem 4.

§9. A Basic Theorem on Removable Singularities

We will consider removable singularities not of individual analytic functions but of complex analytic mappings.

Theorem 5. *Let $\psi(z_1, \cdots, z_k)$ be defined and continuous, or only bounded, in a neighborhood of a point $(a_1, \cdots, a_k)$ and let ψ be analytic in that neighborhood except possibly on an exceptional set E defined by*

$$(59) \quad E: \qquad \Phi(z_1, \cdots, z_k) = 0$$

where Φ *is analytic in a neighborhood of* (a) *with* $\Phi(a) = 0$, $\Phi \not\equiv 0$. *Then* ψ *is analytic in a (complete) neighborhood of the point* (a).

This theorem is not yet sufficiently general for later use in analytic mappings and consequently we will prove the following more general theorem which will be of sufficient generality for our later work. Although these theorems are stated in terms of local conditions the conclusions can be extended to the large.

Theorem 6. *Let* $\Phi(z_1, \cdots z_k)$ *be an analytic function* $(\not\equiv 0)$ *in a neighborhood* U *of a point* $(a_1, \cdots, a_k)$ *in* $(z_1, \cdots, z_k)$*-space, with* $\Phi(a_1, \cdots, a_k) = 0$. *Suppose that* U *is mapped in a 1-1 manner into a point set in* $(w_1, \cdots, w_k)$*-space by real functions* $f_1, \cdots, f_{2k}$ *of the real variables* $x_1, y_1, \cdots, x_k, y_k$, *where* $z_j = x_j + iy_j$ *and*

$$w_1 = f_1 + if_2, \quad \cdots, \quad w_k = f_{2k-1} + if_{2k} \tag{60}$$

and suppose further that each individual f *satisfies a uniform Lipschitz condition of order* λ *in* U *for some* λ *in* $\dfrac{2k-2}{2k-1} < \lambda \leq 1$. *Let* $(b_1, \cdots, b_k)$ *be the image in* (w)*-space of the point* (a). *Then if a function* $G(w_1, \cdots, w_k)$ *is defined and continuous, or only bounded, in all of a neighborhood* V *of* (b) *and analytic in* V *except possibly on those points* E_w *which are images of points* E *satisfying* $\Phi = 0$ *it follows that* G *is analytic in some complete neighborhood of* (b).

If the mapping (60) were assumed to be an analytic mapping with a nonvanishing Jacobian then Theorem 6 would be a consequence of Theorem 5 since in such a case the mapping (60) could be solved for the z's as analytic functions of the w's. But it is precisely in order to be able to handle the case of analytic 1-1 mappings in which no information about the vanishing or nonvanishing of the Jacobian is known that Theorem 6 is needed. But we do point out that since the mapping is continuous and 1-1 it has a continuous inverse, and is thus topological or a homeomorphism. We will state this formally as a lemma.

Lemma 1. *A one-one and continuous mapping* T *of an open subset* U *of a coordinate space* S_n *into a subset of* S_n *is a homeomorphism.*

The proof of this lemma is given more or less explicitly in textbooks on topology.

Returning to our theorems, we are planning to prove Theorem 6. We will need the so-called Weierstrass preparation theorem and some additional material which we are postponing to Chapter IX. At first glance one might assume that our Theorems 5 and 6 are immediate consequences of Theorems 2 and 4 in as much as the "variety" E

defined by (59) is "$(2k\text{-}2)$-dimensional." This is actually so, but not in a manner immediately useful to us. There are theorems in the literature asserting that such a point set can be triangulated into differentiable, or even analytic, so-called simplices of dimensions $\leq 2k - 2$. Even granting the correctness of these theorems they still are not immediately applicable to our purpose. In order to be so, we would need to know that each of the simplices can be imbedded *with its closure* into the *interior* of an enveloping simplex of dimension $\leq 2k - 2$ in such a way that the entire enveloping simplex is "regular." Even if this could be proved, then total demonstration of the triangulation theorem needed would be longer and more involved than our direct proof of our theorems, so that our proof seems to be justified on any count.

The proof (of Theorem 6) will be by means of mathematical induction; the induction will be based upon the number of variables upon which the function Φ depends *explicitly*. First we shall show that the theorem is true whenever the function $\Phi(z_1, \cdots, z_k)$ *depends explicitly upon only one* (*complex*) *variable throughout* U. Let then the hypotheses of Theorem 6 hold and in addition let Φ depend explicitly upon one variable. Without loss in generality we take the points (a) and (b) to be origins in their respective spaces and we denote by z_1 the variable upon which Φ depends explicitly. We write

$$F(z_1) \equiv \Phi(z_1, z_2, \cdots, z_k) \qquad (z) \text{ in } U \tag{61}$$

Then $F(z_1)$ is analytic in a neighborhood of $z_1 = 0$, and $F(0) = 0$, $F(z_1) \not\equiv 0$. Hence there are a positive number ρ and a positive integer m such that

$$F(z_1) = z_1^m \Omega(z_1) \qquad \text{for} \qquad |z_1| < \rho$$

where $\Omega(z_1)$ is analytic and nonvanishing in $|z_1| < \rho$. Thus if ρ' is any positive number less than ρ and such that the polycylinder

$$P(\rho'): \qquad |z_j| < \rho', \qquad j = 1, \cdots, k$$

is contained in U, then the set E, insofar as it lies in $P(\rho')$, consists precisely of the points

$$E: \qquad z_1 = 0, \qquad |z_2| < \rho', \qquad \cdots, \qquad |z_k| < \rho' \tag{62}$$

Under the mapping (60) the points (62) are mapped into the portion of E_w which lies in the image $Q(\rho')$ of $P(\rho')$. (Here we use the fact that (60) is 1-1.) And by Theorem 4 with $n + 2 = 2k$ we conclude that the portion of E_w contained in the neighborhood $Q(\rho')$ of $(w) =$

(0) has the property described by (52). By section 7 we conclude that the function $G(w)$ is analytic in a complete neighborhood of $(w) = (0)$. Thus Theorem 6 holds for the first step of our induction.

Now let p be a fixed integer in

$$1 \leq p < k$$

and for our induction argument let us assume that Theorem 6 holds whenever the function $\Phi(z_1, \cdots, z_k)$ depends explicitly upon at most p complex variables in U. We shall show that the theorem then holds whenever Φ depends upon $(p + 1)$ variables. Let then the hypotheses of Theorem 6 hold and in addition let Φ depend explicitly upon $(p + 1)$ of the variables $z_1, \cdots, z_k$, say $z_1, \cdots, z_{p+1}$. Without loss in generality we again take the points (a) and (b) to be the origins in their respective spaces. We define

$$F(z_1, \cdots, z_{p+1}) \equiv \Phi(z_1, \cdots, z_k)$$

Then $F(z_1, \cdots, z_{p+1})$ is analytic in a neighborhood of $z_1 = \cdots = z_{p+1} = 0$, and $F(0, \cdots, 0) = 0$, $F(z_1, \cdots, z_{p+1}) \not\equiv 0$. We distinguish two cases as follows:

Case A.

$$F(0, \cdots, 0, z_{p+1}) \equiv 0$$

Case B.

$$F(0, \cdots, 0, z_{p+1}) \not\equiv 0$$

In Case A, there is a nonsingular linear homogeneous transformation in the $(z_1, \cdots, z_{p+1})$-space,

$$z_j = \Sigma_{m=1}^{p+1} a_{jm}\zeta_m, \qquad j = 1, \cdots, p + 1 \tag{63}$$

of such a nature that the function

$$F'(\zeta_1, \cdots, \zeta_{p+1}) \equiv F(\Sigma a_{1m}\zeta_m, \cdots, \Sigma a_{p+1,m}\zeta_m)$$

has the property

$$F'(0, \cdots, 0, \zeta_{p+1}) \not\equiv 0$$

(See section 2, Chapter IX.) If we annex to the transformation (63) the transformation

$$z_{p+2} = \zeta_{p+2}, \qquad \cdots, \qquad z_k = \zeta_k \tag{64}$$

then we have a nonsingular homogeneous linear transformation of the original $(z_1, \cdots, z_k)$-space under which the set

E: $$F(z_1, \cdots, z_{p+1}) = 0$$

is transformed into a set

$$E': \qquad F(\zeta_1, \cdots, \zeta_{p+1}) = 0$$

and since the mapping [(63), (64)] from $(z_1, \cdots, z_k)$ to $(\zeta_1, \cdots, \zeta_k)$ has a unique inverse which is linear it follows that the mapping from $(\zeta_1, \cdots, \zeta_k)$ to $(w_1, \cdots, w_k)$ has all the properties of the mapping (60). Thus all the hypotheses of Theorem 6 are valid with $F'(\zeta_1, \cdots, \zeta_{p+1})$ replacing $F(z_1, \cdots, z_{p+1}) \equiv \Phi(z_1, \cdots, z_k)$ and the functions f'_j replacing the f_j of (60), where the f'_j are the transforms of the f_j under [(63), (64)]. Hence it is sufficient to treat Case B.

Under Case B the Weierstrass preparation theorem (see section 1, Chapter IX) states that there is a positive number ρ such that for $|z_1| < \rho, \cdots, |z_{p+1}| < \rho$, the function $F(z_1, \cdots, z_{p+1})$ is expressible in the form

$$F(z_1, \cdots, z_{p+1}) = [z_{p+1}^m + A_1(z_1, \cdots, z_p)z_{p+1}^{m-1} + \cdots + A_m(z_1, \cdots, z_p)]\,\Omega(z_1, \cdots, z_{p+1})$$

where the A's are analytic in $|z_1| < \rho, \cdots, |z_p| < \rho$ and zero at $z_1 = \cdots = z_p = 0$, and Ω is analytic and nonvanishing for $|z_1| < \rho, \cdots, |z_{p+1}| < \rho$. Thus if ρ' is any positive number less than ρ and such that the polycylinder $P(\rho')$: $|z_j| < \rho'$, $j = 1, \cdots, k$, is contained in U and its image $Q(\rho')$ by the transformation (60) is contained in V, then the set E, insofar as it lies in $P(\rho')$ consists precisely of the points $(z_1, \cdots, z_k)$ satisfying

$$(65) \quad \psi(z_1, \cdots, z_{p+1}) \equiv z_{p+1}^m + A_1(z_1, \cdots, z_p)z_{p+1}^{m-1} + \cdots + A_m(z_1, \cdots, z_p) = 0, \ (z) \in P(\rho')$$

Denote by $D(z_1, \cdots, z_p)$ the discriminant of ψ when ψ is viewed as a polynomial in z_{p+1} so that D is obtained by eliminating z_{p+1} from ψ and $\partial\psi/\partial z_{p+1}$. Since D is a polynomial in the coefficients $A_1, \cdots, A_m$ it is an analytic function of $z_1, \cdots, z_p$ in $|z_1| < \rho', \cdots, |z_p| < \rho'$. Again we consider two alternatives, namely

Case 1°.

$$D(z_1, \cdots, z_p) \equiv 0$$

Case 2°.

$$D(z_1, \cdots, z_p) \not\equiv 0$$

A function of the form of $\psi(z_1, \cdots, z_{p+1})$ as described in (65) is called a *distinguished pseudo-polynomial of degree m*. In section 2 of Chapter IX it will be shown that any distinguished pseudo-polynomical of degree m can be decomposed into a product of $\mu (\mu \leq m)$

irreducible factors, say

$$\psi = \psi_1 \cdots \psi_\mu$$

where each ψ_α is an irreducible distinguished psuedo-polynomial (of the same general form as ψ) of degree say n_α with $n_1 + \cdots + n_\mu = m$. The discriminant D of ψ is identically zero if and only if two or more of the factors ψ_α are identical. Thus if Case 1^0 occurs certain of these factors may be discarded leaving the subset consisting of all the distinct ψ_α. The product of these distinct factors defines a distinguished psuedopolynomial ψ^* which has the property that $\psi^* = 0$ and $\psi = 0$ define the same point set, namely E, in a neighborhood of $z_1, \cdots, z_k = 0$. But the discriminant D^* of ψ^* is not identically zero. Hence is sufficient to treat Case 2^0.

Treatment of Case 2^0. In $Q(\rho')$, the image of $P(\rho')$ by (60), we divide the points of E_w into two classes. The first class consists of all points of E_w which are images of points of E (in $P(\rho')$) at which $D(z_1, \cdots, z_p) \neq 0$; the second class consists of all points of E_w which are images of points of E for which $D(z_1, \cdots, z_p) = 0$. We shall first show that $G(w_1, \cdots, w_k)$ is analytic in a neighborhood of every point of the first class. For this purpose let (w^0) be any point of this class. Then (w^0) is the image of a point (z^0) on E for which $D(z_1^0, \cdots, z_p^0) \neq 0$. Since $\psi(z_1^0, \cdots, z_{p+1}^0) = 0$, the nonvanishing of D implies that $\partial\psi/\partial z_{p+1} \neq 0$ at (z^0). Hence by the fundamental existence theorem for implicit functions (see section 4, Chapter II), the equation $\psi = 0$ can be solved for z_{p+1} in a neighborhood of (z^0) yielding

$$z_{p+1} = \varphi(z_1, \cdots, z_p) \tag{66}$$

with $\varphi(z_1^0, \cdots, z_p^0) = z_{p+1}^0$ where φ is an analytic function of $z_1, \cdots, z_p$ in a neighborhood of $(z_1^0, \cdots, z_p^0)$. Hence in a neighborhood of (z^0) the set E is represented by (66). If we decompose φ and $w_1, \cdots, w_k$ into their real and imaginary parts

$$\begin{aligned}\varphi(z_1, \cdots, z_p) &= \alpha(x_1, y_1, \cdots, x_p, y_p) \\ &+ i\beta(x_1, y_1, \cdots, x_p, y_p), \quad w_j = u_j + iv_j, \qquad j = 1, \cdots, k\end{aligned} \tag{67}$$

then the image by (60) of the portion of E lying in a sufficiently small neighborhood of (z^0) can thus be written in the form

$$u_j = f_{2j-1}, \qquad v_j = f_{2j}, \qquad j = 1, \cdots, k$$

where the arguments of f_{2j-1} and f_{2j} are

$$x_1, y_1, \cdots, x_p, y_p, \alpha(x_1, y_1, \cdots, x_p, y_p), \beta(x_1, y_1, \cdots, x_p, y_p), \\ x_{p+2}, y_{p+2}, \cdots, x_k, y_k$$

and this is a mapping of the character required by Theorem 4 (with $n + 2 = 2k$). Hence this portion of E maps into a portion of E_w which has the property (52). The mapping (60) being topological the only portion of E_w in a neighborhood of (w^0) is that which arises from the map of E in a neighborhood of (z^0), and hence by section 7 we conclude that G is analytic in a neighborhood of (w^0).

Thus G is analytic in a neighborhood of every point (w^0) of the first class, and the exceptional set E_w, insofar as it lies in $Q(\rho')$, is reduced to the points of the second class, that is to points on the intersection $E_wE'_w$ where E'_w is the image under (60) of the set E' of points $(z_1, \cdots, z_k)$ for which

$$(68)\quad E': \qquad D(z_1, \cdots, z_p) = 0, \qquad (z) \in P(\rho')$$

We can now apply our induction assumption. The function $G(w_1, \cdots, w_k)$ is defined and either continuous or bounded throughout $Q(\rho')$, is analytic in $Q(\rho')$ except possibly on the set E'_w, and E'_w is the image by (60) of the set (68), where D *depends explicitly only upon* p *variables.* Thus we conclude that G is analytic in a complete neighborhood of $w_1 = \cdots = w_k = 0$. This completes the next step in our induction and hence concludes the proof of Theorem 6.

§10. On Jacobians

We will now prove a significant theorem for Jacobians of functions of several complex variables. For real variables, if the Jacobian is nonvanishing, the transformation is one-one and topological. But the transformation can be topological without the Jacobian being nonvanishing. For example $u = x^3$ is a topological map of $-\infty < x < \infty$ into $-\infty < u < \infty$ and yet the derivative $\dfrac{du}{dx} = 3x^2$ vanishes at $x = 0$. On close scrutiny it turns out that this anomaly is due to the variables being real; indeed we will have the following theorem.

Theorem 7. *Let T be a one-one mapping of an open set U in an analytic coordinate space $\Sigma_{2k}(z)$ into some set V in an analytic coordinate space $\Sigma_{2k}(w)$. In terms of the local coordinates $(z_1, \cdots, z_k)$ in a neighborhood of every point of U let the mapping T into a local coordinate neighborhood $(w, \cdots, w_k)$ in $\Sigma_{2k}(w)$ be given by*

$$(69)\qquad w_j = \varphi_j(z_1, \cdots, z_k), \qquad j = 1, \cdots, k$$

where the functions $\varphi_j(z)$ are analytic functions of $(z_1, \cdots, z_k)$ in the neighborhood in question. Under these conditions the mapping T is a homeomorphism, and in terms of any admissible coordinate system the

Jacobian is nonvanishing throughout,

$$\Phi(z_1, \cdots, z_k) \equiv \frac{\partial(\varphi_1, \cdots, \varphi_k)}{\partial(z_1, \cdots, z_k)} \neq 0 \tag{70}$$

Proof. As we have pointed out in Lemma 1, the mapping T is a homeomorphism; we need accordingly to show only that T has the further property described by (70). For this purpose let p be any point of U with image $q = T_p$, and let T have the form (69) in a sufficiently small neighborhood $N(p)$ of p in U, where the z's and w's are allowable local coordinates valid throughout $N(p)$ and its image $N(q) = TN(p)$ respectively.

We shall first show that the Jacobian cannot vanish identically in $N(p)$. This will depend upon the following known result, which for the sake of completeness we will prove immediately after we complete the remainder of the proof of Theorem 7.

Lemma 2. *Let $\varphi_1(z_1, \cdots, z_k), \cdots, \varphi_k(z_1, \cdots, z_k)$ be analytic in a neighborhood $N(p)$ of some point p. If the Jacobian vanishes identically in $N(p)$, then there exists a relation*

$$\Omega(\varphi_1(z), \cdots, \varphi_k(z)) \equiv 0 \tag{71}$$

in some part of $N(p)$.

Assuming this lemma, we see that the Jacobian (70) cannot vanish identically in $N(p)$, for if it did, then relation (71) would imply that T is not one-one from some part of $N(p)$ into the image part of $N(q)$, thus contradicting the fact that T is a homeomorphism throughout. Thus Φ cannot vanish identically in $N(p)$.

Returning to the proof of Theorem 7 we thus obtain the fact that the relation

$$\Phi(z_1, \cdots, z_k) = 0 \tag{72}$$

is a relation of the type considered in Theorem 6. Therefore relations (69) in $N(p)$ determine an exceptional set E_w in $N(q)$, and in $N(q) - E_w$ the inverse of (69) is given by analytic functions

$$z_j = \psi_j(w_1, \cdots, w_k), \qquad j = 1, \cdots, k \tag{73}$$

But the functions ψ_j are also continuous in $N(q)$, and hence by Theorem 6 they are analytic in all of $N(q)$. Now on substituting (73) into (69) we obtain

$$w_j = \varphi_j(\psi_1(w), \cdots, \psi_k(w))$$

and hence we obtain

$$\frac{\partial(\varphi_1, \cdots, \varphi_k)}{\partial(z_1, \cdots, z_k)} \cdot \frac{\partial(\psi_1, \cdots, \psi_k)}{\partial(w_1, \cdots, w_k)} \equiv 1$$

This is valid not only in $N(q) - E_w$ but in all of $N(q)$ and the second Jacobian is in particular finite. This implies in particular that the first Jacobian is different from zero in all of $N(p)$ as asserted in the theorem.

For the proof of Lemma 2 we note that if the Jacobian vanishes identically then the matrix

$$\frac{\partial(\varphi_1, \cdots \varphi_k)}{\partial(z_1, \cdots, z_k)} \tag{74}$$

has a rank s, $0 \leq s < k$, such that all subdeterminants of rank $> s$ vanish identically, but not all of rank s. If s is zero, then $\partial\varphi_p/\partial z_q \equiv 0$ for all p, q, and hence in this case $\varphi_p \equiv c_p$, which obviously gives relation (71) in the form $\varphi_1 - c_1 \equiv 0$, say. For the case $s > 0$ let

$$\frac{\partial(\varphi_1, \cdots, \varphi_s)}{\partial(z_1, \cdots, z_s)} \not\equiv 0$$

Take a point p_0 at which it is not zero, and assume in new local coordinates that p_0 is the origin. Then the equations

$$\varphi_j(z_1, \cdots, z_s, z_{s+1}, \cdots, z_k) = w_j, \qquad j = 1, \cdots, s \tag{75}$$

can be solved to obtain

$$z_j = \mu_j(z_{s+1}, \cdots, z_k, w_1, \cdots, w_s), \qquad j = 1, \cdots, s \tag{76}$$

where by Theorem 9 of Chapter II the μ's are analytic functions of $z_{s+1}, \cdots, z_k, w_1, \cdots, w_s$ in a neighborhood of $z_{s+1} = \cdots = z_k = 0$, $w_1 = w_1^0, \cdots, w_s = w_s^0$ ($w_p^0 = \varphi_p(0, \cdots, 0)$). On substituting (76) into the $(s+1)$-*st* equation of (69) we obtain

$$\varphi_{s+1}(\mu_1, \cdots, \mu_s; z_{s+1}, \cdots, z_k) - w_{s+1} = 0 \tag{77}$$

This is a relation in

$$z_{s+1}, \cdots, z_k; \qquad w_1, \cdots, w_s, w_{s+1} \tag{78}$$

of the form

$$\varphi^*_{s+1}(z_{s+1}, \cdots, z_k; w_1, \cdots, w_s) - w_{s+1} = 0 \tag{79}$$

which obviously is not fulfilled identically in the variables (78) since w_{s+1} cannot cancel out. However, it follows on the basis of the vanishing of all $(s+1)$-*st* minors of (74) that the variables $z_{s+1}, \cdots, z_k$ do not occur in (79) and thus (79) is a nontrivial relation of the type (71).

This concludes the proof of Lemma 2.

We will draw an interesting conclusion from Theorem 7 but first we state a lemma which we number as Theorem 8.

Theorem 8. *If a sequence of functions $f_n(z_1, \cdots, z_k)$, each of which is nonvanishing in D, converges uniformly in D, then the limit function $f(z_1, \cdots, z_k)$ is either nonvanishing or identically zero.*

This theorem is known for $k = 1$. Hence for $k = 2$ if we suppose that $f(z_1, z_2) = 0$ at say the origin, then we first obtain $f(z_1, 0) = 0$ for all z_1, and then for each z_1^0, we obtain $f(z_1^0, z_2) = 0$ for all z_2. A similar argument applies for any k.

On the basis of Theorem 7 and 8 we will have

Theorem 9. *If a sequence of transformations*

$$S^n: \qquad w_j = f_j^{(n)}(z_1, \cdots, z_k), \qquad j = 1, \cdots, k$$

in a domain D is such that each S^n maps D topologically into some domain $D^{(n)}$, and if in the neighborhood of each point in D, the sequence converges uniformly, then the limit transformation

$$S: \qquad w_j = f_j(z_1, \cdots, z_k)$$

is either again topological in D; or it is degenerate in the sense that its Jacobian vanishes identically.

Proof. Let $\Phi^{(n)}$ be the Jacobian of S^n. Our assumptions imply that $\Phi^{(n)}$ converges uniformly to Φ in a neighborhood of each point of D, and hence by Theorem 8 the latter is either identically zero, which is one possibility, or it is nowhere zero, which is the only other possibility. Now, in the second case S is locally topological, and we still have to show that it is topological in the large. This however follows from Theorem 3 in Chapter III, for if S were to map two different points P, Q into the same point R, then by the latter theorem, for some sufficiently large value of n, the transformation S^n would map a point P^n "near" P and a point Q^n "near" Q both into the same point R^n, contradicting the assumption that S^n is a homeomorphism on D.

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CHAPTER IX

Algebraic Theorems

§1. The Weierstrass Preparation Theorem

If

$$P(w) = \Sigma_{n=0}^{\infty} P_n w^n$$

is an analytic function of the complex variable w in a neighborhood of the origin, if $P(0) = 0$ but $P(w) \not\equiv 0$ so that there exists an index $s > 0$ for which $P_n = 0$ for $n = 0, 1, \cdots, s-1$ but $P_s \neq 0$, then we know that for every function

$$B(w) = \Sigma_{n=0}^{\infty} B_n w^n$$

analytic in a neighborhood of the origin, the quotient

$$Q(w) = \frac{B(w) - \Sigma_0^{s-1} B_n w^n}{P(w)}$$

exists and is analytic in a neighborhood of the origin. In other words, there exists an analytic function $Q(w)$ such that

$$Q(w)P(w) - B(w) = -\Sigma_{n=0}^{s-1} B_n w^n$$

We shall prove an analogue of this for k complex variables.

Lemma 1. *Let*

$$P(z_1, \cdots, z_{k-1}, w) = \Sigma_{n=0}^{\infty} P_n(z) w^n \tag{1}$$

be analytic in a neighborhood of the origin with

$$P_n(0) = 0 \qquad \text{for} \qquad n = 0, 1, \cdots, s-1 \tag{2}$$

but with $P_s(0) \neq 0$. *For convenience take*

$$P_s(0) = 1 \tag{3}$$

Then for every function

$$B(z_1, \cdots, z_{k-1}, w) = \Sigma_{n=0}^{\infty} B_n(z) w^n \tag{4}$$

analytic in a neighborhood of the origin, there exist a function

$$Q(z_1, \cdots, z_{k-1}, w) = \Sigma_{n=0}^{\infty} Q_n(z) w^n \tag{5}$$

analytic in a neighborhood of the origin, and a polynomial in w of degree $s-1$,

$$H(z_1, \cdots, z_{k-1}, w) = \Sigma_{n=0}^{s-1} H_n(z) w^n \tag{6}$$

with

$$(7) \qquad H_n(0) = B_n(0), \qquad n = 0, 1, \cdots, s-1$$

and with $H_n(z)$, $n = 0, 1, \cdots, s-1$, *analytic in a neighborhood of the origin, such that*

$$(8) \qquad Q(z, w)P(z, w) - B(z, w) = H(z, w)$$

The function Q *and the "polynomial"* H *are uniquely determined by* (8) *together with the requirements that they be analytic in a neighborhood of the origin, that* H *be of degree* $s-1$ *in* w *and that* (7) *hold for the coefficients of* H.

For later use (in section 2) it is convenient to separate the formal part of this lemma from its analytic (that is, convergence) part. For this purpose we consider formal power-series as in equation (1) of Chapter I. We may write a formal power-series $P(z_1, \cdots, z_{k-1}, w)$ in the form (1) where the $P_n(z)$ are formal power-series in $z_1, \cdots, z_{k-1}$. For example, if

$$P(z_1, \cdots, z_{k-1}, w) = \Sigma^{\infty}_{n_1, \cdots, n_{k-1}, n=0} a_{n_1, \cdots, n_{k-1}, n} z_1^{n_1} \cdots z_{k-1}^{n_{k-1}} w^n$$

then the coefficients $P_n(z)$ of (1) would be

$$P_n(z) \equiv \Sigma^{\infty}_{n_1, \cdots, n_{k-1}=0} a_{n_1 \cdots n_{k-1}, n} z_1^{n_1} \cdots z_{k-1}^{n_{k-1}}$$

If we write $P_n(z) \equiv 0$ we will mean that $a_{n_1, \cdots, n_{k-1}, n} = 0$ for $n_1, \cdots, n_{k-1} = 0, 1, 2, \cdots$. We may also write (formally) $P_n(0) = 0$ to mean $a_{0 \cdots 0n} = 0$, or $P_s(0) = 1$ to mean $a_{0 \cdots 0s} = 1$. We shall on occasions write a formal series in $z_1, \cdots, z_{k-1}$, such as $P_n(z)$, in a formal series of the form

$$P_n(z) = \Sigma^{\infty}_{\mu=0} p_{n\mu}(z)$$

where the $p_{n\mu}(z)$ are homogeneous polynomials in $z_1, \cdots, z_{k-1}$ of degree μ. Obviously $P_n(z) \equiv 0$ is equivalent with $p_{n\mu}(z) \equiv 0$ for $\mu = 0, 1, 2, \cdots$, and, since the $p_{n\mu}$ are polynomials, the relations $p_{n\mu}(z) \equiv 0$ are not only "formal" but also "analytic" identities. With these conventions we proceed to prove the following lemma which differs but little from Lemma 1 itself.

Lemma 1a. *Let* (1) *be a formal power-series and let* (2) *and* (3) *hold. Then for every formal power series* (4) *there exist a formal power-series* (5) *and a polynomial* (6) *in* w *of degree* $s-1$ *with* (7) *holding and with its coefficients* $H_n(z)$ *formal power-series in* $z_1, \cdots, z_{k-1}$, *such that* (8) *holds.*

The series Q and the polynomial H are uniquely determined (as formal power-series) by (8) together with the requirements that H be of degree $s - 1$ in w and that (7) hold.

Proof of Lemma 1a. If (8) is to hold we must have

$$(9) \qquad \Sigma_{\mu=0}^{m} Q_\mu(z) P_{m-\mu}(z) - B_m(z) \equiv 0 \qquad \text{for} \qquad m \geq s$$

Furthermore, if we can show that there exists a unique series (5) such that (9) holds then we will have our desired conclusion. For we may use (8) to define H uniquely and clearly H will be of the required form. Thus our proof of Lemma 1a is reduced to determining (uniquely) a formal series (5) for which (9) holds.

If we write

$$(10) \quad P_m(z) \equiv \Sigma_{n=0}^{\infty} p_{mn}(z), \qquad Q_m(z) \equiv \Sigma_{n=0}^{\infty} q_{mn}(z), \qquad B_m(z) \equiv \Sigma_{n=0}^{\infty} b_{mn}(z)$$

where p_{mn}, q_{mn} and b_{mn} are homogeneous polynomials of degree n in $z_1, \cdots, z_{k-1}$, then (9) is equivalent with

$$(11) \quad \Sigma_{\mu=0}^{m} \Sigma_{\nu=0}^{n} q_{\mu\nu}(z) p_{m-\mu,n-\nu}(z) - b_{mn}(z) \equiv 0 \qquad \text{for} \qquad m \geq s, \qquad n \geq 0$$

By (2) and (3) and the fact that $p_{n0}(z)$ is constant for each $n = 0, 1, 2, \cdots$,

$$(12) \quad p_{m0}(z) \equiv 0 \qquad \text{for} \qquad m = 0, 1, \cdots, s-1; \qquad p_{s0}(z) = 1$$

so that

$$\begin{aligned}(13) \quad & \Sigma_{\mu=0}^{m} \Sigma_{\nu=0}^{n} q_{\mu\nu}(z) p_{m-\mu,n-\nu}(z) \\ & \equiv \Sigma_{\mu=0}^{m} \Sigma_{\nu=0}^{n-1} q_{\mu\nu}(z) p_{m-\mu,n-\nu}(z) + \Sigma_{\mu=0}^{m} q_{\mu n}(z) p_{m-\mu,0}(z) \\ & \equiv \Sigma_{\mu=0}^{m} \Sigma_{\nu=0}^{n-1} q_{\mu\nu}(z) p_{m-\mu,n-\nu}(z) + \Sigma_{\mu=0}^{m-s-1} q_{\mu n}(z) p_{m-\mu,0}(z) + q_{m-s,n}(z)\end{aligned}$$

We understand that the sum $\Sigma_{\nu=0}^{n-1}$ is to be replaced by zero if $n = 0$, and similarly for $\Sigma_{\mu=0}^{m-s-1}$ *when* $m = s$. Thus (11) becomes

$$(14) \quad q_{m-s,n}(z) \equiv b_{mn}(z) - \Sigma_{\mu=0}^{m} \Sigma_{\nu=0}^{n-1} q_{\mu\nu}(z) p_{m-\mu,n-\nu}(z) - \Sigma_{\mu=0}^{m-s-1} q_{\mu n}(z) p_{m-\mu,0}(z)$$

for $m \geq s$, $n \geq 0$. On replacing m by $m + s$, (14) becomes

$$(15) \quad q_{mn}(z) \equiv b_{m+s,n}(z) - \Sigma_{\mu=0}^{m+s} \Sigma_{\nu=0}^{n-1} q_{\mu\nu}(z) p_{m+s-\mu,n-\nu}(z) + \Sigma_{\mu=0}^{m-1} q_{\mu n}(z) p_{m+s-\mu,0}(z)$$

for $m \geq 0$, $n \geq 0$. In particular, if $m = n = 0$, (15) yields

$$(16) \qquad q_{00}(z) \equiv b_{s0}(z)$$

We shall now show that the q_{mn} can be defined from (15) by recursion. For this purpose we introduce the index

(17) $$g(m,n) = m + (s + 1)n$$

Now for all $q_{\mu\nu}$ occurring on the right-hand side of (15) we see that $g(\mu, \nu)$ is greatest when either $\mu = m + s$, $\nu = n - 1$, or when $\mu = m - 1$, $\nu = n$, and therefore

$$\begin{aligned} g(\mu, \nu) &\leq \max\,[m + s + (s + 1)(n - 1),\ m - 1 + (s + 1)n] \\ &= m + (s + 1)n - 1 = g(m, n) - 1 \end{aligned}$$

Hence for all $q_{\mu\nu}$ occurring on the right-hand side of (15), $g(\mu, \nu) < g(m, n)$. Therefore relation (15) suffices to define the $q_{mn}(z)$ uniquely in terms of those $q_{\mu\nu}(z)$ for which $g(\mu, \nu) < g(m, n)$.

This concludes the proof of Lemma 1a. For the proof of Lemma 1 we have to show that the function $Q(z, w)$ defined with these q's (by means of (10) and (5)) is analytic in a neighborhood of the origin whenever P and B have this property. The analyticity of H in a neighborhood of the origin will follow from (8).

From the analyticity of $P(z, w)$ and $B(z, w)$ we know that there exist positive numbers M, ρ, σ such that

$$|P_m(z)| < \frac{M}{\sigma^m}, \qquad |B_m(z)| < \frac{M}{\sigma^m} \qquad \text{for} \qquad m = 0, 1, 2, \cdots$$

and for $|z_1| < \rho, \cdots, |z_{k-1}| < \rho$, and hence

$$|p_{mn}(z)| < \frac{M}{\sigma^m}, \qquad |b_{mn}(z)| < \frac{M}{\sigma^m} \qquad \text{for} \qquad m, n = 0, 1, 2, \cdots;$$
$$|z_1| < \rho, \cdots, |z_{k-1}| < \rho$$

Since by a linear transformation on w we can send the circle $|w| < \sigma$ into the circle $|w| < 1$, it involves no loss in generality to assume $\sigma = 1$, or

(18) $$|p_{mn}(z)| < M, \qquad |b_{mn}(z)| < M, \qquad m, n = 0, 1, 2, \cdots$$

for $z_1, \cdots, z_{k-1}$ in the multi-cylinder A_ρ,

(19) $$A_\rho: \qquad |z_1| < \rho, \qquad \cdots, \qquad |z_{k-1}| < \rho$$

We propose to prove the existence of quantities K, C and c such that

(20) $$|q_{mn}(z)| < KC^m c^n, \qquad m, n = 0, 1, 2, \cdots; z \text{ in } A_\rho$$

First take $K > 1$, $C > 1$, $c > 1$ and K sufficiently large so that

$$|q_{00}| < KC^0c^0$$

Clearly this can be done. We shall then prove (20) by induction on the index $g(m, n)$ defined in (17). For an arbitrary m_0, n_0 with $g(m_0, n_0) > 0$ we assume (20) to be true for every $q_{\mu\nu}$ for which $g(\mu, \nu) < g(m_0, n_0)$, and we shall then prove (20) for each q_{mn} for which $g(m, n) = g(m_0, n_0)$. For any such q_{mn} we have by (15), (18) and our induction assumption

$$\begin{aligned}|q_{mn}(z)| &< M + M\Sigma_{\mu=0}^{m+s}\Sigma_{\nu=0}^{n-1}KC^{\mu}c^{\nu} + M\Sigma_{\mu=0}^{m-1}KC^{\mu}c^{n}\\ &< M + MK\frac{C^{m+s+1}}{C-1}\cdot\frac{c^n}{c-1} + MKc^n\frac{C^m}{C-1}\end{aligned}$$

for z in A_ρ. Our task is to choose C and c (with $C > 1$, $c > 1$) so that this latter expression does not exceed KC^mc^n, or what is the same, so that

$$\frac{M}{KC^mc^n} + \frac{MC^{s+1}}{(C-1)(c-1)} + \frac{M}{C-1} \leq 1 \tag{21}$$

If we take $C > 3M + 1$, $c > C^{s+1} + 1$, it is clear that (21) is satisfied. This completes our induction proof, and hence (20) holds for $m, n = 0, 1, 2, \cdots$, and z in A_ρ. But this means that the series

$$Q(z, w) = \Sigma_{m,n=0}^{\infty} q_{m,n}(z)w^m$$

converges absolutely and uniformly for (z, w) in any closed set interior to $[|z_1| < \rho/c, \cdots, |z_{k-1}| < \rho/c, |w| < \sigma]$ and hence Q is analytic in a neighborhood of the origin. This concludes the proof of Lemma 1.

We use these results for the special case

$$B(z, w) \equiv w^s \tag{22}$$

Then by Lemma 1a we know that there exist a formal series $Q(z, w)$ in $z_1, \cdots, z_{k-1}, w$ and formal series $H_0(z), \cdots, H_{s-1}(z)$ in $z_1, \cdots, z_{k-1}$ with $H_j(0) = 0$ for which

$$Q(z, w)P(z, w) \equiv w^s + H_{s-1}(z)w^{s-1} + \cdots + H_0(z)$$

the series Q and $H_0, \cdots, H_{s-1}$ being uniquely determined. Furthermore, since $q_{00} = 1$ $(= b_{s0})$ (see (16) and (22)), we have $Q(0, 0) = 1$. Now it is easily seen (and will also be shown near the beginning of the next section) that any formal power-series whose constant term is nonvanishing has a (unique) reciprocal with a nonvanishing constant term. Thus

$$\frac{1}{Q(z, w)} = \Omega(z, w)$$

where Ω is a formal power-series in $z_1, \cdots, z_{k-1}, w$ with $\Omega(0, 0) \neq 0$.

This yields the Weierstrass preparation theorem for formal power-series. Also by Lemma 1, if P is analytic in a neighborhood of the origin, then so are Q, Ω and the H's. This then yields the customary (analytic) formulation of the preparation theorem.

Theorem 1. *(The Weierstrass preparation theorem.) Let $P(z_1, \cdots, z_{k-1}, w)$ be analytic in a neighborhood of the origin, with*

$$P(0, \cdots, 0, 0) = 0, \qquad P(0, \cdots, 0, w) \not\equiv 0 \tag{23}$$

Then there exist a neighborhood U of the origin and a function $\Omega(z_1, \cdots, z_{k-1}, w)$ which is analytic and nonvanishing in U such that throughout U, P can be expressed in the form

$$P(z, w) \equiv (w^s + H_{s-1}(z)w^{s-1} + \cdots + H_0(z))\,\Omega(z, w) \tag{24}$$

where the H's are analytic in a neighborhood of $z_1 = \cdots = z_{k-1} = 0$

and

$$H_j(0, \cdots, 0) = 0 \qquad j = 0, 1, \cdots s - 1 \tag{25}$$

If the integer s is taken to be the order of the zero of $P(0, \cdots, 0, w)$ at $w = 0$, then the functions Ω, $H_0, \cdots, H_{s-1}$ are uniquely determined.

We may now also formulate the theorem for formal power-series. It is, however, more convenient to delay the formulation until the next section where it will be given as Theorem 2.

We will conclude this section with an alternative proof of the preparation theorem. This proof will be based on analytic considerations and will be independent of the one just given.

Proof of Theorem 1 (from analytic considerations). Since $P(z, w)$ is analytic in a neighborhood of the origin and since $P(0, w) \not\equiv 0$, it follows that there are an $\epsilon > 0$ and a $\sigma > 0$ such that $P(z, w)$ is analytic for $|z_1| \leq \sigma, \cdots, |z_k| \leq \sigma$, $|w| \leq \epsilon$ and such that $|P(0, w)| \geq \alpha > 0$ for $|w| = \epsilon$. Moreover, $P(0, w)$ is expressible as a (convergent) power-series of the form

$$P(0, w) = P_s w^s + P_{s+1} w^{s+1} + \cdots, \qquad |w| \leq \epsilon \tag{26}$$

where $s > 0$ and $P_s \neq 0$. Now there is a subneighborhood A_ρ: $|z_1| < \rho, \cdots, |z_{k-1}| < \rho$ with $\rho < \sigma$, such that in it

$$|P(z, w) - P(0, w)| < \alpha \qquad \text{for} \qquad |w| = \epsilon$$

But Rouche's theorem states that if $F_1(w)$ and $F_2(w)$ are analytic in $|w| \leq \epsilon$ and if $|F_1(w)| < |F_2(w)|$ for $|w| = \epsilon$, then $F_1(w) + F_2(w)$ has the same number of zeros in $|w| < \epsilon$ as $F_2(w)$. Putting $F_1(w) =$

$P(z, w) - P(0, w)$ (for any fixed z in the neighborhood A_ρ), and $F_2(w) = P(0, w)$, it follows that the function $P(z, w)$, as a function of w, has s zeros for $|w| < \epsilon$, z in A_ρ.

Let us take a fixed $(z_1^0, \cdots, z_{k-1}^0)$ in the neighborhood A_ρ and let the values of w for which $P(z^0, w) = 0$ be

$$w_1(z^0), \cdots, w_s(z^0)$$

Now as is well known, if $p(w)$ and $\varphi(w)$ are analytic functions of a complex variable w in $|w| \leq \epsilon$ and if p has s zeros $w_1, \cdots, w_s$ in $|w| < \epsilon$ and none on $|w| = \epsilon$, then

$$\frac{1}{2\pi i} \int_{|w|=\epsilon} \varphi(w) \frac{p'(w)}{p(w)} \, dw = \varphi(w_1) + \cdots + \varphi(w_s)$$

Specializing $\varphi(w)$ as w^r, we obtain for our function $P(z^0, w)$ the relation

$$w_1^r(z^0) + \cdots + w_s^r(z^0) = \frac{1}{2\pi i} \int_{|w|=\epsilon} w^r \frac{P_w(z^0, w)}{P(z^0, w)} \, dw$$

where $P_w(z^0, w) = \partial P(z^0, w)/\partial w$. But the integrand is analytic in (z^0, w) for $|w| = \epsilon$, z^0 in A_ρ since $P(z^0, w) \neq 0$ for $|w| = \epsilon$, and consequently $[w_1^r(z) + \cdots + w_s^r(z)]$ is analytic in $(z_1, \cdots, z_{k-1})$ in A_ρ. Since the elementary symmetric functions are polynomials in the sums of powers we obtain the result that

$$(27) \qquad [w - w_1(z)] \cdots [w - w_s(z)]$$

can be written in the form

$$w^s + H_{s-1}(z) w^{s-1} + \cdots + H_0(z) \equiv \pi(z, w)$$

where the H's are analytic in A_ρ. Since $P(0, w)$ has an s-fold zero at $w = 0$ (see (26)), each $H_p(z)$, $p = 0, 1, \cdots, s - 1$, vanishes at $z_1 = \cdots = z_{k-1} = 0$.

Clearly the function $\Omega(z, w)$ defined by

$$\Omega(z, w) = \frac{P(z, w)}{\pi(z, w)}$$

does not vanish at the origin. The uniqueness properties of π and Ω are also clearly verified. It remains to show that Ω is analytic in a neighborhood of the origin. Now for fixed z^0 in A_ρ, the function

$$\Omega(z^0, w) = \frac{P(z^0, w)}{\pi(z^0, w)}$$

is analytic in w in a neighborhood of $w = 0$. We must show that for w^0 in $|w| < \epsilon$, the function $\Omega(z, w^0)$ is analytic in z in a neighborhood of the origin in the z-space. First we have

$$\Omega(z, w^0) = \frac{1}{2\pi i} \int_{|\zeta|=\epsilon} \frac{\Omega(z, \zeta)}{\zeta - w^0} d\zeta \tag{28}$$

On the contour, $\pi(z, \zeta) \neq 0$ for z in a neighborhood of the origin, and hence $\Omega(z, \zeta)$ is analytic in $(z_1, \cdots, z_{k-1}, \zeta)$. Consequently $\Omega(z, w^0)$ is analytic in $(z_1, \cdots, z_{k-1})$ in a neighborhood of the origin. Hence by Hartog's theorem (Theorem 4, Chapter VII) it follows that $\Omega(z, w)$ is analytic in $(z_1, \cdots, z_{k-1}, w)$ in a neighborhood of the origin. This completes the (analytic) proof of Theorem 1.

§2. DISTINGUISHED POLYNOMIALS

In section 1 of Chapter I we saw that the totality of power-series in k variables $z_1, \cdots, z_k$ forms a (commutative) ring, multiplication being defined purely formally without regard to convergence. We shall call this ring I_k.

If P is an element of I_k then we can write it as

$$P = p_0 + p_1 + p_2 + \cdots \tag{29}$$

where p_j is a homogeneous polynomial of degree j in $z_1, \cdots, z_k$. If p_n is the first p not identically zero, we say that n is the *degree* of the element. Thus for example, an element of degree zero is one with a nonvanishing constant term, and an element of degree greater than zero is one with a vanishing constant term.

Now in section 1 of Chapter I we saw that the ring I_k has no null-divisors, that is $P \cdot Q = 0$ for P and Q elements of I_k means that either $P = 0$ or $Q = 0$ (or both). It is quite easy to see that I_k has *units*, a unit being defined as an element P of I_k which has an inverse Q under multiplication, $P \cdot Q = 1$. In fact, *the set of units is identical with the set of all elements of I_k of degree zero*. To see this we first note that if P is a unit, then on writing it in the form (29) and its inverse Q in the form $Q = q_0 + q_1 + q_2 + \cdots$ we have by definition of multiplication in I_k

$$p_0 q_0 = 1, \qquad \Sigma_{j=0}^{m} p_j q_{m-j} = 0, \qquad m = 1, 2, \cdots \tag{30}$$

Thus in particular we see that $p_0 \neq 0$ and hence P is of degree zero. Conversely, let P be any element of I_k of degree zero. Write P in the form (29) where $p_0 \neq 0$. Then by recursion we may determine

a unique element $Q = q_0 + q_1 + \cdots$ such that $P \cdot Q = 1$; we do this by means of the relations (30). Thus P is a unit.

Since our ring has units and no null-divisors it is in fact a *domain of integrity*. We will use this fact later.

Lemma 2. *If P is an element of degree $s \geq 1$, there exists a nonsingular linear transformation such that in the transform P' there is a term of the form $cz_1'^s$ where c is a nonzero constant.*

Remark. When this is the case, and no lower power of z_1' occurs with a constant coefficient, we say that P' is *regular* with respect to z_1' with *degree s*.

Proof of Lemma 2. We have

$$P = p_s + p_{s+1} + \cdots$$

Let

$$p_s = \Sigma_{s_1+\cdots+s_k=s} a_{s_1\cdots s_k} z_1^{s_1} \cdots z_k^{s_k}$$

and put

$$z_h = \Sigma_{j=1}^{k} b_h^j z_j', \qquad h = 1, \cdots, k$$

In p_s' the coefficient of $z_1'^s$ will be

$$\Sigma_{s_1+\cdots+s_k=s} a_{s_1\cdots s_k} (b_1^1)^{s_1} \cdots (b_n^1)^{s_k} \tag{31}$$

Since $p_s(z_1, \cdots, z_k) \not\equiv 0$ this means that $p_s(b_1^1, \cdots, b_k^1) \not\equiv 0$ and hence there must exist values of $b_1^1, \cdots, b_k^1$ (of which some will necessarily differ from zero) such that the term (31) will be nonvanishing. With these values of $b_1^1, \cdots, b_k^1$ we can obviously choose b_j^h, ($j = 1, \cdots, k$, $h = 2, \cdots k$) such that the determinant of the b's is different from zero, and we shall then have a nonsingular linear transformation which makes P' regular with respect to z_1', with degree s.

Lemma 1a can be restated as follows. If P is regular with respect to z_k with degree $s \geq 1$, and if B is any element of I_k, then there exist two elements Q and H such that

$$QP = B + H$$

where

$$H = A_1 z_k^{s-1} + \cdots + A_s$$

and where the A's are elements of I_{k-1}.

The Weierstrass preparation theorem can be stated formally as follows.

Theorem 2. *If P is an element of I_k which is regular with respect to z_k with degree $s \geq 1$, then*

$$P = (z_k^s + A_1 z_k^{s-1} + \cdots + A_s)\Omega \tag{32}$$

where the A's are elements of I_{k-1} each of degree greater than zero, and Ω is an element of I_k of degree zero.

The elements Ω and $A_1, \cdots, A_s$ are uniquely determined.

Definition of equivalence in I_k. If P and Q are any two elements of I_k so related that $P = QE$, where E is a unit, then we say that P and Q are equivalent.

Definition of a distinguished polynomial. If π is an element of I_k which can be written in the form

$$A_0 z_k^s + A_1 z_k^{s-1} + \cdots + A_s \tag{33}$$

where the A's are elements of I_{k-1}, we say that π belongs to $I_{k-1}[z]$; that is, the ring of polynomials with coefficients in I_{k-1}. The element π is called *distinguished* in $I_{k-1}[z_k]$ if $A_0 = 1$, and if each A_j, $j = 1, \cdots, s$, is of degree greater than zero.

We may thus restate Theorem 2 (the Weierstrass preparation theorem) in the following form.

Corollary 1. *Every element P of I_k which is regular with respect to z_k is equivalent to a distinguished polynomial in $I_{k-1}[z_k]$ of the same degree.*

We now state and prove the following lemma, mainly for purposes of reference.

Lemma 3. *If $P = Q \cdot R$, where P, Q and R are elements of I_k, and if P is regular with respect to z_j then either*

a) *one of the elements Q or R is a unit and the other is regular with respect to z_j,* or

b) *both Q and R are regular with respect to z_j.*

Proof. By definition, an element $T = t_s + t_{s+1} + \cdots$ $(s \geq 1)$ is regular with respect to z_j if and only if

$$t_s(0, \cdots, 0, z_j, 0, \cdots, 0) \not\equiv 0 \tag{34}$$

Now denote by α, β, γ the degrees of P, Q, and R respectively. Then clearly $\alpha = \beta + \gamma$, $\alpha > 0$, $\beta \geq 0$, $\gamma \geq 0$. Write

$$P = p_\alpha + p_{\alpha+1} + \cdots, \qquad Q = q_\beta + q_{\beta+1} + \cdots,$$
$$R = r_\gamma + r_{\gamma+1} + \cdots$$

If either β or γ is zero, say $\beta = 0$, then Q is a unit and $\gamma = \alpha$ and in this case $p_\alpha = q_0 r_\alpha (q_0 \neq 0)$ and since p_α has the property (34) it follows that r_α does also and thus r_α is regular with respect to z_j. If, on the other hand, $\beta\gamma \neq 0$, the $p_\alpha = q_\beta r_\gamma$ and we see that both q_β and r_γ must have the property (34). This yields the lemma.

Definition of reducibility. If $P = Q \cdot R$, where P, Q and R are elements of I_k and neither Q nor R is a unit, we say that P is *reducible* in I_k. If no such relation exists P is called *irreducible* in I_k.

Lemma 4. *If P is distinguished in $I_{k-1}[z_k]$, and is reducible in I_k, it is reducible in $I_{k-1}[z_k]$.*

Proof. Let $P = Q \cdot R$ where Q and R are nonunit elements of I_k. Since P is regular with respect to z_k it follows from Lemma 3 that Q and R are both regular with respect to z_k, and hence by Corollary 1 each is equivalent to a distinguished polynomial in $I_{k-1}[z_k]$. Let $Q = q \cdot E'$, $R = r \cdot E''$ where the small letters denote distinguished elements of $I_{k-1}[z_k]$ and the E's denote units in I_k. Then we have $P = q \cdot r \cdot E^*$. But clearly $q \cdot r$ is also distinguished, and since the preparation theorem ensures uniqueness in the representation as the product of a distinguished polynomial and a unit, we deduce that $E^* = 1$ and $P = q \cdot r$. Thus we have shown that P is reducible in $I_{k-1}[z_k]$.

Theorem 3. *Any element P of I_k is uniquely decomposable into irreducible factors, to within equivalence.*

Proof. The proof is inductive; we assume that the theorem is valid in I_{j-1} for some j in the range $1, \cdots, k$, and show that it follows for I_j. Since it is trivially true in I_0 (the underlying field in which every element $\neq 0$ is a unit) it will follow for I_k.

Consider any element P of I_j. If P is of degree zero it is a unit and hence already decomposed to within equivalence. Hence let P be of degree greater than zero. Factorization is manifestly left invariant by nonsingular linear transformations. Accordingly we may assume without loss in generality that P is regular with respect to z_j (see Lemma 2). Now by Corollary 1, P is equivalent to a distinguished polynomial, f, in $I_{j-1}[z_j]$.

But there is a theorem in algebra (see for example reference 8 on page 203) which asserts that if R is a domain of integrity admitting unique decomposition, then $R[z]$ has the same property. Now by this and our induction assumption we see that f is uniquely decomposible into irreducible elements of $I_{j-1}[z_j]$. But Lemma 4 showed that any decomposition of f in I_j gave factors equivalent to elements of $I_{j-1}[z_j]$. Hence two decompositions of f must give the same set of equivalent polynomial factors and hence must be equivalent decompositions. This completes the induction and yields the theorem.

For the remainder of this section we will be dealing with convergent series, that is, with elements of I_k or $I_{k-1}[z_k]$ which are analytic in a neighborhood of the origin $z_1 = \cdots = z_k = 0$.

Theorem 4. *If f_1 and f_2 are elements of $I_{k-1}[z_k]$, each analytic in a neighborhood of the origin, if f_2 is distinguished and irreducible, and if f_1 vanishes for every set of values for which f_2 does, then f_2 is a factor of f_1.*

Proof. We use the fact that if f and g are elements of $I_{k-1}[z_k]$ with no common factor, then there exists a linear combination of them with coefficients in $I_{k-1}[z_k]$ which lies entirely in I_{k-1} and which is not (identically) zero. This follows at once from the Euclidean algorithm which can be shown to hold in $I_{k-1}[z_k]$.

Suppose f_2 is not a factor of f_1; since f_2 is irreducible this means that f_1 and f_2 have no common factor and hence there exist elements λ_1, λ_2 of $I_{k-1}[z_k]$ such that

$$\lambda_1 f_1 + \lambda_2 f_2 = P \tag{35}$$

where P is a nonzero element of I_{k-1}. Now by hypothesis, f_1 is zero whenever f_2 is. But since f_2 is distinguished, and analytic in some neighborhood N of the origin, we know that for any set of values $(\zeta_1, \cdots, \zeta_{k-1})$ in a sufficiently small neighborhood of $z_1 = \cdots = z_{k-1} = 0$, there must exist a ζ_k, with $(\zeta_1, \cdots, \zeta_k)$ in N, for which $f_2 = 0$. Thus the element P of (35) is zero throughout N' and hence P is the zero element which is a contradiction. Hence f_2 must be a factor of f_1.

Corollary 2. *If f and g are elements of I_k, each analytic in a neighborhood of the origin, if g is irreducible and if f vanishes wherever g does, then g is a factor of f.*

Proof. Obviously we may assume that f and g are not units. By Lemma 2 we may make a nonsingular linear transformation such that the transform $f'g'$ of the product fg is regular with respect to z_k'. Then by Lemma 3 both f' and g' are regular with respect to z_k' and hence by Corollary 1 each is equivalent to a distinguished polynomial in $I_{k-1}[z_k]$. Theorem 4 then applies to yield the corollary.

Corollary 3. *If f and g are elements of I_k, each analytic in a neighborhood of the origin, and if f vanishes wherever g does, then every irreducible factor of g is a factor of f.*

§3. CHARACTERISTIC MANIFOLDS

Let $f(w, z_2, \cdots, z_k)$ be an analytic function, not identically zero, which is irreducible, and equal to zero at a given point which for the sake of simplicity we take to be the origin. Suppose further that $f(w, 0, \cdots, 0) \not\equiv 0$. (By Lemma 2 this can always be brought about by a nonsingular linear transformation.) Then by the prepa-

ration theorem, Theorem 1, there are a neighborhood

$$(36) \qquad U: \quad |w| < d_1, \qquad |z_j| < d_j, \qquad j = 2, 3, \cdots, k$$

and a function $\Omega(w, z)$ analytic and nonvanishing in U such that $f = \pi\Omega$ throughout U, where $\pi(w, z)$ is a distinguished polynomial

$$\pi(w, z) = w^s + C_1(z)w^{s-1} + \cdots + C^s(z)$$

The coefficients $C_1, \cdots, C_s$ are analytic functions of $z_2, \cdots, z_k$ for $|z_j| < d_j$, $j = 2, \cdots, k$, and each vanishes at $z_2 = \cdots = z_k = 0$.

We shall be concerned with the manifold defined by $f = 0$. Since $\Omega \neq 0$ throughout U, it follows that throughout U the manifold $f = 0$ is identical with that defined by $\pi = 0$, which we now proceed to investigate.

The discriminant $D(z_2, \cdots, z_k)$ of the function $\pi(w, z)$ viewed as a polynomial in w, is a polynomial in the coefficients $C_\nu(z_2, \cdots, z_k)$ and hence is analytic in the projection Z of U on the $(z_2, \cdots, z_k)$-space,

$$Z: \qquad |z_j| < d_j, \qquad j = 2, \cdots, k$$

If the degree s of $\pi(w, z)$ as polynomial in w is unity, then $D(z_2, \cdots, z_k) = 1$; if on the other hand $s > 1$, then D vanishes at the origin. But since by assumption, $f(w, z_2, \cdots, z_k)$ is irreducible it follows that π is irreducible. By a general theorem, which applies to our situation, the discriminant vanishes (identically, that is) if and only if the polynomial has two equal irreducible factors. Hence $D \not\equiv 0$.

Let z^0: $(z_2^0, \cdots, z_k^0)$ be a point of Z distinct from the origin at which $D(z_2, \cdots, z_k) \neq 0$. At each point z in a sufficiently small neighborhood of z^0 there will be s distinct roots of $\pi = 0$ which we denote by

$$(37) \qquad w_1 = \varphi_1(z), \cdots, w_s = \varphi_s(z)$$

Corresponding to these we shall have s distinct points of U: $P_1, \cdots, P_s$, all possessing the same projection z, and such that π vanishes at each of them.

The fundamental existence theorem for implicit functions tells us that the roots as functions of $z_2, \cdots, z_k$ are differentiable up to the same order as the original implicit function; hence in our case they are analytic functions of $z_2, \cdots, z_k$ within some polycylinder $P(z^0, r)$ about z^0.

Now denote by N the set of points $(z_2, \cdots, z_k)$ in Z at which $D(z_2, \cdots, z_k) \neq 0$. We shall show that N is connected and that

each of the s functions $\varphi_1, \cdots, \varphi_s$ of (37) can be analytically continued along a suitable closed path into each of the others. First we show that N is connected. For this purpose we prove the following lemma.

Lemma 5. *If $f(w, z)$ is a distinguished polynomial in $I_{k-1}[w]$, whether irreducible or not, then the set V consisting of those points of U at which $f(w, z_2, \cdots, z_k) \neq 0$ is a connected set.*

Proof. In any two-dimensional plane

$$(38) \qquad |w| < \infty, \qquad z_2 = a_2, \cdots, z_k = a_k$$

with $(a_2, \cdots, a_k)$ a fixed point of Z, there are only a finite number of points of U at which $f(w, z) = 0$. Denote by $p(a)$ the portion of the plane (38) lying in U, that is, the portion for which $|w| < d_1$. Obviously, $p(a)$ is connected, and since $p(z)$ has only a finite number of points in $U - V$, it follows that the intersection $p(a) \cdot V$ is connected.

Now suppose that V were not connected. Then there would exist two sets V_1 and V_2 such that $V_1 + V_2 = V$ and such that no limit point of V_1 lies in V_2 and vice versa. Since $p(a) \cdot V$ is connected it must be entirely either in V_1 or in V_2. Thus the points of Z can be divided into two classes Z_1 and Z_2, the set Z_j $(j = 1, 2)$ consisting of those points $(a_2, \cdots, a_k)$ of Z for which the intersection $p(a) \cdot V$ lies in V_j.

But Z is a connected set; and hence there exists a sequence of points $a^1 = \{a_2^1, \cdots, a_k^1\}$, $a^2 = \{a_2^2, \cdots, a_k^2\}$, $\cdots$ of Z_1 having a limit point a in Z_2. In each of the planes $p(a^1), p(a^2), \cdots$ there are only a finite number of points of U at which $f(w, z) = 0$, i.e. only a finite number of points of $U - V$. Hence the projections of all such points of all the planes $p(a^1), p(a^2), \cdots$ upon the plane $p(a)$ give a countable set. Let P be a point of $p(a)$ not belonging to this countable set, and let $P_1, P_2, \cdots$ be its projections on $p(a^1), p(a^2), \cdots$. Then $P_1, P_2, \cdots$ belong to V_1 but their limit point P belongs to V_2, which furnishes a contradiction to the supposition that V is not connected. This demonstrates connectedness in the weakest topological sense.

To demonstrate arcwise connectedness, we have only to remark that from the continuity of $f(w, z)$ there exists, about every point of U at which $f \neq 0$, a spherical or multi-cylindrical neighborhood in which $f \neq 0$, so that V is locally connected. It is a known result in topology that these two properties together give arcwise connectedness.

This yields Lemma 5.

Lemma 5 tells us that if we have a function f which is analytic in a neighborhood of a point, which vanishes at the point, and which is irreducible, then there exists a subneighborhood U of the point such that in U the set defined by $f \neq 0$ is a connected set. We want to apply this to the discriminant $D(z_2, \cdots, z_k)$ of the distinguished polynomial $\pi(w, z)$, in a neighborhood of the origin. If the degree s of π is unity, then as we have noted $D(z) \equiv 1$ and obviously the set $D(z) \neq 0$ is a connected set. If, on the other hand, $s > 1$, then as we have also noted, $D(z)$ vanishes at the origin. A linear transformation will carry $D(z)$ into the product of a distinguished polynomial in $I_{k-2}(z_k)$ with a nonvanishing factor, and Lemma 5 will imply that the set N (in the place of the set V of the lemma) is a connected one.

Since the set N defined by $D(z_2, \cdots, z_k) \neq 0$ is a connected set we may consider for any point z^0 of N all closed paths in N beginning and terminating with z^0. Further, consider the totality of functional elements which arise at z^0 by continuing $\varphi_1, \cdots, \varphi_s$ around these closed paths any number of times. They must be identical with $\varphi_1, \cdots, \varphi_s$ in some order, for they will continue to satisfy the equation $\pi(w, z) = 0$. As we remarked earlier, we shall show that any one of them, say φ_1, will give rise to *all* the others. If this were not so, we should have φ_1 giving rise only to $\varphi_1, \cdots, \varphi_m$, $m < s$, on being continued in every possible way and returning to z^0. Then the set $\varphi_1, \cdots, \varphi_m$ would permute among themselves on further continuation; it would not be possible for two elements, say φ_λ and φ_μ to give rise to the same element φ_ν, since at all points of the closed paths $D(z) \neq 0$. Hence the symmetric functions

$$\begin{aligned} B_1(z) &= \varphi_1 + \cdots + \varphi_m, \\ B_2(z) &= \varphi_1\varphi_2 + \cdots + \varphi_{m-1}\varphi_m \\ B_m(z) &= \varphi_1\varphi_2 \cdots \varphi_m \end{aligned}$$

would be unchanged under such continuations, and likewise therefore the polynomial

$$Q(w, z) = w^m - B_1(z)w^{m-1} + \cdots + (-1)^m B_m(z)$$

Now each $B_\mu(z_2, \cdots, z_k)$, $\mu = 1, \cdots, m$, is single-valued and analytic in N, that is, throughout the domain Z except on the set where $D(z_2, \cdots, z_k) = 0$. And even when $D = 0$, the φ's and hence the B's are defined, and bounded in every interior subdomain. Hence by Theorem 5, Chapter VIII each B_μ is analytic throughout Z, that is, even when $D(z) = 0$. Hence the distinguished polynomial

$Q(w, z)$ is analytic throughout the domain (36). Also it vanishes at each point at which π does, and hence by Theorem 4, π is a factor of Q, which means that $m > s$. Thus if $f(w, z)$ is irreducible we have $m = s$ and each of the functions $\varphi_1, \cdots, \varphi_s$ goes into all the others after analytic continuation along closed paths in N.

We now define the characteristic manifold of $f(w, z)$. The manifold M defined by

$$f(w, z_2, \cdots, z_k) = 0, \qquad D(z_2, \cdots, z_k) \neq 0$$

possesses the property that at each of its points it is possible to express one of the variables (namely w) as an analytic function of the others. In general this will not be true of the points for which $D(z) = 0$; but if there are points of $f = 0$, $D = 0$ with this property, we will denote them collectively by M'. We define the totality of points $M + M'$ as the *characteristic* manifold of $f = 0$.

§4. A Remark on Algebraic Functions

We will now point out what from our approach is a very simple if not trivial remark but which appears in older literature as part of a statement on "Abelian functions."

We first observe that the preceding sections of this chapter include all algebraic prerequisites which were referred to in the previous chapter on removable singularities and that therefore we may conversely apply the results of Chapter VIII. Take a domain D in $(z_1, \cdots, z_k)$-space, or better yet, an arbitrary complex space S in k complex variables. A point set E in S will now be termed "exceptional" if corresponding to any point P in S there exist a coordinate neighborhood N of P in S and an analytic function $\varphi(z_1, \cdots, z_k)$ in N such that the points EN are all included among the points z for which $\varphi = 0$. By the argument of section 3, $S - E$ is again a (connected) space. Now we assume that for a fixed integer r, corresponding to any point P in $S - E$, there exist r functional elements

$$f_1^P(z), \cdots, f_r^P(z) \tag{39}$$

such that each element of each point can be obtained from any other element at any other point by analytic continuation along a suitable path in $S - E$. More precisely, we assume that there exist an r-sheeted covering space T over $S - E$ (without "singularities" or "relative boundary points") and an analytic function on T for which (39) are r different functional elements on r points of T having the same projection onto $S - E$. As in section 3 it follows that these

functionals are solutions of an irreducible equation

$$f(w, z) \equiv w^r + A_1(z)w^{r-1} + \cdots + A_r(z) = 0 \tag{40}$$

where the $A_\rho(z)$ are symmetric functions of (39) and analytic in $S - E$.

Now comes the crucial assumption. We assume that all functional elements have a common bound. Indeed, if S is not a compact set, it suffices to make the following assumption. Corresponding to any point Q in S, there exists a neighborhood N of Q such that in the parts of T projecting onto $(S - E)N$ the absolute value of our function $f(w, z)$ has a bound. By Theorem 5 of Chapter VIII it now follows that the $A_\rho(z)$ are analytic in $S \cdot N$ and thus in S. In other words, *if a function f is r-valued and bounded on $S - E$, it is an algebraic function on all of S.*

§5. Rational and Algebraic Functions on Products of Domains

An analytic function $f(t', \cdots, t^{(k)})$ in a domain D of real or complex variables $t = (t', \cdots, t^{(k)})$ is called rational if there exist polynomials $P(t)$, $Q(t)$ such that

$$P(t)f(t) + Q(t) = 0 \tag{41}$$

We do not require that $P(t)$ shall be $\not\equiv 0$ throughout and thus we cannot insist on writing $f(t)$ in the form $-\dfrac{P(t)}{Q(t)}$ as a quotient of two polynomials. If relation (41) holds in some subdomain D_0 of D then by analytic continuation it extends to all of D. It is obvious that sums and products of rational functions are again rational, and if analytic functions $f_1(t), \cdots, f_n(t)$ satisfy a relation

$$r_1(t)f_1(t) + \cdots + r_n(t)f_n(t) \equiv 0$$

with rational coefficients $r_\nu(t)$ then they also satisfy such a relation with polynomial coefficients.

The theorem to follow will be based on the following lemma.

Lemma 6. *If A is a domain in z-space, and B a domain in w-space, if $F_1(z, w), \cdots, F_N(z, w)$ are analytic functions in the product domain $A \times B$, not all $\equiv 0$, if every $F_n(z, w)$ is a rational function in w for every point z in A, and if they satisfy a relation*

$$c_1(w)F_1(z, w) + \cdots + c_N(w)F_N(z, w) \equiv 0 \tag{42}$$

with arbitrary (not necessarily analytic) functions $c_n(w)$ for which

$$|c_1(w)|^2 + \cdots + |c_N(w)|^2 > 0 \tag{43}$$

then there exist polynomials $C_1(w), \cdots, C_N(w)$, *not all* $\equiv 0$, *such that*

$$C_1(w)F_1(z, w) + \cdots + C_N(w)F_N(z, w) \equiv 0 \tag{44}$$

Proof. We write relation (42) for N values $z_1, \cdots, z_N$, thus obtaining a system of N linear homogenous equations in the quantities $c_n(w)$. Because of (43), the determinant

$$D(z, w) = \begin{vmatrix} F_1(z_1, w) & \cdots & F_N(z_1, w) \\ \cdots & \cdots & \cdots \\ F_1(z_N, w) & \cdots & F_N(z_N, w) \end{vmatrix}$$

must vanish identically in $w \varepsilon B$; $z_1 \varepsilon A$, $z_2 \varepsilon A$, $\cdots$, $z_N \varepsilon A$. We now replace the letter z_N by z, and we develop the determinant in terms of its last row. This leads to a relation

$$\Sigma_{n=1}^{N} C_n(z_1, \cdots, z_{N-1}; w)F_n(z, w) \equiv 0 \tag{45}$$

in which the functions

$$C_n(z_1, \cdots, z_{N-1}; w) \tag{46}$$

are, but for $\pm$ signs, the $(N - 1)$-dimensional determinants of the matrix

$$\left\| \begin{matrix} F_1(z_1, w) & \cdots & F_N(z_1, w) \\ \cdots & \cdots & \cdots \\ F_1(z_{N-1}, w) & \cdots & F_N(z_{N-1}, w) \end{matrix} \right\|$$

We now assume first that not all functions (46) vanish identically in all variables. Hence there exist numerical values $z_1 = a_1, \cdots, z_{N-1} = a_{N-1}$, such that the functions

$$C_n(w) \equiv C_n(a_1, \cdots, a_{N-1}; w) \tag{47}$$

do not all vanish identically in w, and for such numerical values (45) reduces to (44), since the functions $C_n(w)$, being rational combinations of the functions $F_m(a_n, w)$, are rational functions in w by the hypothesis of our lemma.

If however all functions (46) vanish identically there exists a principal minor

$$\Delta = \begin{vmatrix} F_{\alpha_1}(z_{\alpha_1}, w) & \cdots & F_{\alpha_m}(z_{\alpha_1}, w) \\ \cdots & \cdots & \cdots \\ F_{\alpha_1}(z_{\alpha_m}, w) & \cdots & F_{\alpha_m}(z_{\alpha_m}, w) \end{vmatrix}$$

of the determinant D which vanishes identically but which has at least one subdeterminant of order $(m - 1)$ which does not vanish iden-

tically. By the result just proved there exists a relation

$$\Sigma_{\mu=1}^{m} C_{\alpha_\mu}(w) F_{\alpha_\mu}(z, w) \equiv 0$$

and by inserting vanishing coefficients $C_n(w)$ we obtain relation (42).

We can now proceed to our theorem.

Theorem 5. *If A and B are domains in two sets of variables $z = (z', \cdots, z^{(n)})$, $w = (w', \cdots, w^{(l)})$, and if an analytic function $f(z, w)$ in the product space $A \times B$ is a rational function in z for each value w and a rational function in w for each value of z, then $f(z, w)$ is rational in (z, w).*

On the surface, this theorem strongly resembles the theorem of Hartogs (for complex variables) that analyticity in each variable implies analyticity in all variables. However, in substance the present theorem is totally different, being preponderantly algebraic with only a thin layer of analysis.

Denoting the sequence of monomials $\{z^{(1)n_1} \cdots z^{(k)n_k}\}$, in any arrangement, by $p_1(z)$, $p_2(z)$, $\cdots$ we first have, for each w, a relation

$$(\Sigma_{\mu=1}^{m} a_\mu(w) p_\mu(z)) f(z, w) + \Sigma_{\nu=1}^{n} b_\nu(w) p_\nu(z) \equiv 0 \tag{48}$$

with

$$\Sigma_{\mu=1}^{m} |a_\mu(w)|^2 + \Sigma_{\nu=1}^{n} |b_\nu(w)|^2 > 0 \tag{49_1}$$

For the moment we make the admissible normalization:

$$\Sigma_{\mu=1}^{m} |a_\mu(w)|^2 + \Sigma_{\nu=1}^{n} |b_\nu(w)|^2 = 1 \tag{49_2}$$

Now, let $\{w_s\}$, $s = 1, 2, \cdots$, be any sequence of points in B, for which relation (48) is available with the same indices (m, n), and let the sequence be convergent to a point w_0. On account of (49_2), there exists a sub-sequence $\{w_{s_r}\}$, for which the limits

$$\lim_{r\to\infty} a_\mu(w_{s_r}) = a_\mu, \qquad \lim_{r\to\infty} b_\nu(w_{s_r}) = b_\nu$$

exist, and because of continuity of $f(z, w)$, relation (48) with those limit values will be valid for w_0. In other words, if $B_{m,n}$ is the set of those points in B, for which a relation (48) with the given values (m, n) is available, then $B_{m,n}$ is closed relative to B. Since $\Sigma_{m,n=1}^{\infty} B_{m,n} = B$, and B is open, some of the sets $B_{m,n}$ must have an interior B_0. Replacing B by B_0, we have now reached the stage of m, n being fixed integers independent of w, and this leads directly to an application of Lemma 6. Putting $N = m + n$, $F_\mu(z, w) = p_\mu(z) f(z, w)$, $\mu = 1, \cdots, m$, and $F_{m+\nu}(z, w) = p_\nu(z)$, $\nu = 1, \cdots, n$, we conclude by the lemma that there exist *polynomials* $a_\mu(w)$, $b_\nu(w)$ for which (48) and

(49_1) hold, and this is precisely our contention that $f(z, w)$ is rational in (z, w).

Lemma 6 admits other conclusions. We call an analytic function $f(t)$ *algebraic* of degree $\leq s$ if there exist polynomials $P_0(t), \cdots, P_s(t)$ such that $\Sigma_{r=0}^{s} P_r(t)(f(t))^r \equiv 0$. A function is rational if it is algebraic of degree 1. Now, if an analytic function $f(z, w)$ is algebraic in z for each w, then we have a relation

$$(50) \qquad \Sigma_{\mu=1}^{m} \Sigma_{r=0}^{s} a_{\mu r}(w) p_\mu(z)(f(z, w))^r \equiv 0$$

and for some subdomain B_0 of B this relation is available for the same (m, s). We now assume that for each a in A, the function $f(w) \equiv f(a, w)$ is also algebraic in w, meaning that it satisfies an equation of the form

$$(51) \qquad C_0(w)f^p + C_1(w)f^{p-1} + \cdots + C_p(w) \equiv 0$$

the coefficients C_ρ being polynomials in w. We do not inquire into the manner in which the integer p or the polynomials $C_\rho(w)$ depend upon a; also the highest coefficient $C_0(w)$, which of course is not identically zero, may very well vanish at some points of B. But we remind the reader that by our explicit assumption the function $f(z, w)$ is analytic in $A \times B$. Now, returning to relation (50), we are going to apply our lemma to the functions

$$(52) \qquad F_n(z, w) \equiv H_{\mu,r}(z, w) \equiv p_\mu(z)(f(z, w))^r$$

Now it follows from Lemma 6, or rather from the proof of the lemma, that there exists a fixed pair of indices (m, s) such that relation (50) is available for coefficients $a_{\mu r}(w)$ each of which is a polynomial in terms which arise from (52) by specializing the value z. In other words, each $a_{\mu r}(w)$ is a polynomial of terms each of which is an algebraic function in w, and thus each $a_{\mu r}(w)$ itself is an algebraic function in w. The way we have obtained equation (50), the coefficients $p_\mu(z)$ are polynomials in z but the $a_{\mu r}(w)$ are only algebraic functions. But, by a known theorem in algebra we may hence obtain another relation (50) in which both $p_\mu(z)$ and $a_{\mu r}(w)$ are polynomials in their variables.

However the degree may have to be increased. But, if all $f(a, w)$ are rational in w, the degree s need not be altered. Thus we obtain the following theorem, the second half of which includes Theorem 5 for the special value $s = 1$.

Theorem 6. *If $f(z, w)$ is algebraic in z for each w and algebraic in w for each z then $f(z, w)$ is algebraic in (z, w). If $f(z, w)$ is algebraic in z*

for each w, of a degree $\leq s$, where s is independent of w, and if $f(z, w)$ is rational in w for each z, then $f(z,w)$ is algebraic of degree $\leq s$ in (z, w).

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CHAPTER X

Local Analytic Varieties

§1. Introduction. The Basis Condition

This chapter is based on a paper by Walther Rückert, Zum Eliminationsproblem der Potenzreihenideale, *Math. Ann.* Vol. 107 (1933), pp. 259–281. The object is to examine the local properties of the zeros of a number of analytic functions, and to arrive at a parametric representation of an irreducible analytic manifold, which had previously been familiar to analysts, by the use of primarily algebraic methods.

We will recall the notation and theorems already given which are relevant to the discussion.

I_n is the set of all power-series of n complex variables convergent in some preassigned domain about the origin; we know that I_n forms a ring.

An element of I_n which has the form $z_n^r + a_1(z_1, \cdots, z_{n-1})z_n^{r-1} + \cdots + a_r(z_1, \cdots, z_{n-1})$ where $a_1(z_1, \cdots, z_{n-1})$ are elements of I_{n-1} vanishing at the origin is called *a distinguished polynomial in* z_n.

An element of I_n, which is not identically zero when $z_1, \cdots, z_{n-1}$ are put equal to zero, is said to be *regular in* z_n.

Two elements of I_n are considered *equivalent* if one may be obtained from the other by multiplication by an element not vanishing at the origin. Such an element is a *unit* of the ring.

The Weierstrass preparation theorem (Theorem 1, Chapter IX) tells us that any element of I_n which is regular in z_n is equivalent to a distinguished polynomial in z_n.

It was proved in Chapter VIII that factorization in I_n is unique to within equivalence; and also that any element of I_n can be transformed by a linear transformation into one regular in I_n.

We now give the following algebraic theorem concerning I_n:

Theorem 1. *I_n fulfills the basis condition; that is, every ideal in I_n has a finite basis.*

The proof goes by induction; we assume that it is true for I_{n-1}. Since clearly it holds for I_0, the underlying number field, the proof will be complete when the step of the induction is shown.

If the ideal $\mathfrak{a}$ contains no element regular in z_n, make a linear transformation to give one element Q having this property. Clearly

this transformation does not affect the issue. Let z_n^r be the leading term of the distinguished polynomial to which Q is equivalent. Then we know that any element of I_n, and therefore any element P of the ideal $\mathfrak{a}$ is expressible as $QF + P^*$, where P^* is a polynomial in z_n of degree less than r, and with coefficients in I_{n-1}. If now we can find a basis for all P^*'s obtained in this way, this basis with Q added will give us a basis for $\mathfrak{a}$.

Considering this set of P^*'s, it is easy to see that the set of coefficients of z_n^{r-1} in them forms an ideal in I_{n-1}; from the hypothesis of the induction this ideal has a basis $p_1, \cdots, p_\lambda$. Take one P^* having p_1 as leading coefficient and call it P_1^*, and so on. Now we see that having $Q, P_1^*, \cdots, P_\lambda^*$ as a partial basis, any element of $\mathfrak{a}$ can be expressed as the sum of multiples of these plus a polynomial in z_n of degree less than $r - 1$. But again the coefficients, in this residual set, of z_n^{r-2} form a ideal in I_{n-1}, and we get a further addition to the basis of $P_{\lambda+1}^*, \cdots, P_{\lambda+\mu}^*$. Proceeding in this way we eventually construct a finite basis for the ideal $\mathfrak{a}$, since we have a finite number of steps and at each step we add a finite number of elements to the basis. Then applying the induction successively we obtain the required theorem.

§2. Analytic Manifolds at a Point

An analytic manifold at the origin is defined to be the common zeros, in some domain D containing the origin, of a finite number of nonunit elements of I_n; we consider D to be the domain in which elements must converge to be admitted as elements of I_n. We have a set of functions each vanishing at the origin; each function gives a point set of zeros in D, and we take that point set which is common to all the functions defining the manifold.

If $f_1, \cdots, f_p$ are the analytic functions in question, it is clear that $\Sigma\lambda_i f_i$ will also be an analytic function having the manifold contained in its zeros: that is, any element of the ideal $\{f_1, \cdots, f_p\}$ constructed on $f_1, \cdots, f_p$ as a basis, will vanish over the manifold.

But we can consider an ideal connected with this manifold which may be larger than this, namely, the set of all elements of I_n vanishing on it; this is an ideal, and it obviously contains the previous one. We shall call this the *proper ideal of the manifold.*

If $g_1^{\lambda_1} g_2^{\lambda_2} \cdots g_k^{\lambda_k}$ is an element of I_n vanishing over some manifold, then $g_1 g_2 \cdots g_k$ will also vanish over it, since it has the same zeros as the other. In fact if $g_1^{\lambda} \cdots g_k^{\lambda_k}$ is an element of the proper ideal of a manifold, so is $g_1 \cdots g_k$. An equivalent statement is that, if f^p is an element of a proper ideal, then f itself also is. It will be seen

later that if an ideal has this property, then it must be the proper ideal of the manifold it defines. We say that an ideal defines the manifold of the common zeros of all its elements.

A simple example of an ideal which is not the proper ideal of its manifold can be given in I_2: namely $\{z_1^2\}$; the zeros of this are the points $z_1 = 0$, and the ideal of this manifold is the ideal $\{z_1\}$.

§3. IRREDUCIBLE MANIFOLDS

If we have two ideals with bases $(f_1, \cdots, f_k)$, $(g_1, \cdots, g_h)$, then their sum is the ideal with basis $(f_1, \cdots, f_k, g_1, \cdots, g_h)$. The manifold of this ideal is the common zeros of $f_1, \cdots, f_k, g_1, \cdots, g_h$, that is the common points of the manifolds of the two original ideals. So the manifold of the sum of two ideals is the intersection of their manifolds. If $(f_1, \cdots, f_k)$, $(g_1, \cdots, g_h)$ are the proper ideals of their manifolds it does not follow that their sum is the proper ideal of the intersection manifold. As an example in I_2 we can take $\{z_1 + z_2^2\}$ and $\{z_1 - z_2^2\}$; the sum of these ideals is the ideal $\{z_1, z_2^2\}$. This is not a proper ideal because it contains z_2^2 as an element but not z_2, whereas the two original ideals were both proper to their manifolds.

If we take two manifolds M_1 and M_2, and their proper ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$, then the intersection of the ideals is the proper ideal of the point set sum of M_1 and M_2. For if f covers $M_1 + M_2$ it covers both M_1 and M_2, and is therefore an element of both proper ideals, $\mathfrak{a}_1$, and $\mathfrak{a}_2$, and consequently of their intersection. Equally any element of the intersection is an element of each and covers both M_1 and M_2.

An analytic manifold is said to be *irreducible* at the origin if it cannot be expressed there as the sum of two analytic manifolds neither of which contains the other. An ideal is called *prime* if, from fg being an element we can deduce that either f or g is an element. We now obtain the important theorem:

Theorem 2. *The necessary and sufficient condition that a manifold M be irreducible is that its proper ideal be prime.*

Proof. (a). If M is irreducible, the proper ideal, $\mathfrak{p}$, is prime. For, let $\mathfrak{p}$ not be prime; then we find f, g such that $fg \in \mathfrak{p}$, $f, g \notin \mathfrak{p}$. Then the ideals $\mathfrak{p} + (f)$, $\mathfrak{p} + (g)$ are larger ideals than $\mathfrak{p}$, and consequently the manifolds they define are each different from M, and will in fact be properly contained in it. But their intersection is precisely $\mathfrak{p}$. For let $p_1 + \lambda f$ be an element of the first which is equal to $p_2 + \mu g$ of the second, where p_1 and p_2 are elements of $\mathfrak{p}$. Then $\lambda f = p + \mu g$, where $p \in \mathfrak{p}$. Multiplying by μg, $\lambda\mu fg = p' + \mu^2 g^2$.

But $fg \varepsilon p$, therefore $(\lambda\mu fg - p')$ also εp and so $\mu^2 g^2 \varepsilon p$. But p is the proper ideal of its manifold, and so since $\mu^2 g^2$ is an element, μg also is; therefore the original element common to both ideals, $(p_2 + \mu g)$, is necessarily an element of p, or their intersection is p. Therefore the sum of the manifolds of $p + (f)$ and $p + (g)$, is the manifold of p, and M has been expressed as the sum of two manifolds, both properly contained in it. But this is impossible since we assumed that it was irreducible.

(b). If p is the proper ideal of M and is prime, then M is irreducible. Suppose $M = M_1 + M_2$, where neither contains the other. Then, if p_1 and p_2 are their proper ideals, each contains elements not contained in the other. Let p_1 be in p_1 and not in p_2, and p_2 be in p_2 and not in p_1. Then $p_1 p_2$ lies in both and therefore in their intersection, but neither p_1 nor p_2 lies in the intersection. But the intersection is p, the proper ideal of the sum of M_1 and M_2, and what has been shown contradicts the hypothesis that it is prime.

This completes the proof that the necessary and sufficient condition for M to be irreducible is that its proper ideal be prime.

We now come to the first main theorem.

Theorem 3. *An analytic manifold at a point can be uniquely decomposed as the sum of irreducible manifolds.*

This follows from a well-known algebraic theorem concerning rings fulfilling the basis condition (see van der Waerden, *Moderne Algebra*, Vol. 2, §83). In any ring we can define primary ideals, which are generalizations of prime ideals. An ideal is prime if from $fg \varepsilon p$, $f \notin p$, we can deduce that $g \varepsilon p$. An ideal p is defined as *primary* if the deduction is only that we can find a finite integer ρ, such that $g^\rho \varepsilon p$. Given any primary ideal, we can construct a prime ideal belonging to it which contains it; it will consist of all elements f of the ring such that some $f^\rho \varepsilon p$.

Now the theorem about rings fulfilling the basis condition is that any ideal can be expressed as the intersection of a finite number of primary ideals, $q_1, \cdots, q_s$; and that if we stipulate that no q is superfluous in the intersection, and that to each q corresponds a different prime p, then this decomposition is unique as far as the primes belonging to the q's are concerned. Now since the manifold of a primary ideal is identical with the manifold of the prime ideal belonging to it, this shows that we can express any analytic manifold as the sum of a finite number of manifolds defined by prime ideals and that this can be done in only one way. We know that if the proper ideal is prime the manifold is irreducible and if we also know that a prime

ideal must be the proper ideal of the manifold it defines, we have proved the theorem. This will be shown later, and is not, in point of fact, essential to the theorem if we start by decomposing the proper ideal of the manifold in question.

The second main theorem concerns the local parametric representation of an irreducible analytic manifold; it states that, by proper choice of coordinate axes, the manifold can be expressed in terms of a subset $z_1, \cdots, z_s$ of the coordinates as parameters, the number of parameters being half the dimension of the manifold. We make use of an algebraic function w of $z_1, \cdots, z_s$ defined by

$$g(w) \equiv w^\rho + A_1 w^{\rho-1} + \cdots + A_\rho = 0$$

where the A_i are analytic functions of $z_1, \cdots, z_s$. Then the remaining variables $z_{s+1}, \cdots, z_n$ are given by algebraic functions of the form

$$z_{s+1} = \frac{P_{s+1}(w)}{D}$$

$$\cdots$$

$$z_n = \frac{P_n(w)}{D}$$

where $P_i(w)$ are polynomials in w of degree less than ρ and having as coefficients analytic functions of the parametric variables, and where D is the discriminant of $g(w)$. In this way any set of values of $z_1, \cdots, z_s$ for which $D \neq 0$ gives ρ values for w, any one of which gives unique values to $z_{s+1}, \cdots, z_n$.

We begin by defining a *regular ideal;* an ideal $\mathfrak{a}$ is said to be regular with $z_1, \cdots, z_k$ as independent variables, if

(i) $\mathfrak{a}$ contains no element independent of $z_{k+1}, \cdots, z_n$;

(ii) if $h > k$, $\mathfrak{a}$ contains some element independent of $z_{h+1}, \cdots, z_n$ which is regular in z_h.

Theorem 4. *Any ideal can be made regular by means of a non-singular linear transformation of coordinates.*

Take an element of $\mathfrak{a}$ and transform the coordinates, if necessary, to make it regular in z_n. Now if no element of $\mathfrak{a}$ is independent of z_n, then the ideal is regular, with z_n as the independent variable. If not take again an element independent of z_n and transform $z_1, \cdots, z_{n-1}$ to make it regular in z_{n-1}. By a repeated use of this process we can clearly continue until there is no term independent of $z_1, \cdots, z_k$, and the ideal will be regular.

This means that, given an analytic manifold, we can choose the coordinate directions so that its proper ideal is regular.

Now at each step an element is picked out and made regular in z_s; by the Weierstrass preparation theorem it is equivalent to a distinguished polynomial in z_s, and the coefficients will be functions of $z_1, \cdots, z_{s-1}$. Obviously this polynomial will also be an element of the ideal; we will denote it by $F_s(z_s)$.

We now consider the case of a prime ideal which has been made regular. F_s can be taken as irreducible as a polynomial in z_s; for if it were not, some irreducible factor of it would have to belong to the ideal $\mathfrak{p}$, and we could use this instead, as it would also be distinguished in z_s.

Theorem 5. *If $\mathfrak{p}$ is a regular prime ideal, the residue $I_n/\mathfrak{p}$ is isomorphic to $I_k[\eta_{p+1}, \cdots, \eta_k]$ where η_s is an element algebraic over I_k. η_s will be the residue class containing z_s, and the algebraic equation in η_s will be distinguished.*

Since we have $F_n = z_n^r + A_1 z_n^{r-1} + \cdots + A_r$ and F_n is an element of $\mathfrak{p}$, we can consider the corresponding equation between residue classes in $I_n/\mathfrak{p}$, and we get $0 = \eta_n^r + \{A_1\}\eta_n^{r-1} + \cdots + \{A_r\}$. We can represent by R_p the ring of residue classes of $I_n/\mathfrak{p}$ which contains elements independent of $z_{p+1}, \cdots, z_n$. Then $\{A_i\}$ in the above equation is an element of R_{n-1} and we have that $I_n/\mathfrak{p} = R_{n-1}[\eta_n]$, where the algebraic equation in η_n is the above; it will be irreducible since F_n is.

Now making use of F_{n-1} we show that $R_{n-1} = R_{n-2}[\eta_{n-1}]$. Eventually we reach the equation $R_{k+1} = R_k[\eta_{k+1}]$. But R_k is isomorphic to I_k; for if two different elements of I_k belonged to the same residue class, their difference would be a nonzero element of $\mathfrak{p}$ dependent only on $(z_1, \cdots, z_k)$, which is contrary to hypothesis.

So we have that $I_n/\mathfrak{p} \simeq I_k[\eta_{k+1}][\eta_{k+2}] \cdots [\eta_n]$. But it is well known that successive integral algebraic extensions of a ring are equivalent to simultaneous integral extensions, from the principle of the transitivity of integral dependence. It will also follow that the equations for η_s having coefficients in I_k will be distinguished, since each of the equations in the successive extensions is distinguished. We may then write $I_n/\mathfrak{p} \simeq I_k[\eta_{k+1}, \cdots, \eta_n]$ and the theorem is proved.

We have $I_n/\mathfrak{p} \simeq I_k[\eta_{k+1}, \cdots, \eta_n]$. Let us consider the quotient field of this ring; it can be formed since $\mathfrak{p}$ is prime and the ring is a domain of integrity. If Λ_k is the field of all rational analytic functions in $z_1, \cdots, z_k$, it is known that the quotient field $\simeq \Lambda_k[\eta_{k+1}, \cdots, \eta_n]$;

that is, it consists of polynomials in $\eta_{k+1}, \cdots, \eta_n$ with coefficients in Λ_k.

It is now easy to show that $\mathfrak{p}$ must be the proper ideal of its manifold. If it were not there would be a larger ideal defining the same manifold; let g be an element of this, but not of $\mathfrak{p}$. Then the residue class of g, $\{g\}$, is an element of $I_k[\eta_{k+1}, \cdots, \eta_n]$ and of $\Lambda_k[\eta_{k+1}, \cdots, \eta_n]$; since it is not zero it has an inverse $\{h\}$, such that $\{h\}\{g\} = 1$. $\{h\}$ will be a polynomial in $\eta_{k+1}, \cdots, \eta_n$ with coefficients in Λ_k: take the common denominator of these coefficients as p_k, then $\{h\} = P(\eta_{k+1}, \cdots, \eta_n)/P_k$, where $P(\eta)$ is an element of $I_k[\eta_{k+1}, \cdots, \eta_n]$, which is not zero. That is $\{g\}P(\eta) = p_k$ which is independent of $(z_{k+1}, \cdots, z_n)$. This means that any ideal containing $\mathfrak{p}$ and g, contains p_k, a nonzero function of $(z_1, \cdots, z_k)$. But this immediately shows that the manifold of any ideal containing $\mathfrak{p}$ is definitely smaller than that of $\mathfrak{p}$, and that $\mathfrak{p}$ is therefore the proper ideal of its manifold. In passing it is worth stating that from this it follows that if $\mathfrak{a}$ is such that if it contains f^ρ, it contains f, then it also is the proper ideal of its manifold. This is equivalent to asserting the truth of the Hilbert *Nullstellensatz* for analytic functions: namely, that if f vanishes over the common zeros of $f_1, \cdots, f_n$, then there is some positive integer ρ such that

$$f^\rho = \lambda_1 f_1 + \cdots + \lambda_n f_n$$

To return to the quotient field $\Lambda_k[\eta_{k+1}, \cdots, \eta_n]$. Since η_s satisfies an irreducible equation, the extension is separable, and consequently can be generated by the single extension by a primitive element w. (See van der Waerden, *Moderne Algebra*, Vol. 1, p. 120.) We can take $w = d_{k+1}\eta_{k+1} + \cdots + d_n\eta_n$ where d_s lies in I_k.

It is not hard to show that the equation satisfied by w will also be distinguished; let it be

$$G(w) = w^\rho + a_1 w^{\rho-1} + \cdots + a^\rho = 0$$

We will designate the discriminant of G by $D(z_1, \cdots, z_k)$; since G is irreducible this will not be identically zero. It is a known theorem of algebra that every element of $\Lambda_k[w]$ which is integral over I_k, is expressible as $\alpha \equiv \Sigma_{i=0}^{\rho-1} b_i \dfrac{w_i}{D}$, where b_i is in I_k. (See van der Waerden, *Moderne Algebra*, Vol. 2, p. 93.) But it was shown above that η_s was integrally dependent on I_k, so that we have

$$\eta_s \equiv \Sigma_{i=0}^{\rho-1} \frac{b_i^s w^i}{D}$$

Theorem 6. *If we take a set of values for* $z_1, \cdots, z_k$ *for which* $D(z_1, \cdots, z_k)$ *does not vanish, then substituting in* $G(w)$ *we get* ρ *different solutions for* w*; choosing any one of these solutions we may find the corresponding values of* $\eta_{k+1}, \cdots, \eta_n$*. Then every such set of values* $(\zeta_1, \cdots, \zeta_k, \eta_{k+1}, \cdots, \eta_n)$ *gives a zero of the ideal* $\mathfrak{p}$*, and every zero of* $\mathfrak{p}$ *for which* $D \neq 0$ *is reached by this process.*

We need a preliminary lemma.

Since the equation for η_s in terms of $(z_1, \cdots, z_k)$ is distinguished, we can choose values $(a_1, \cdots, a_k)$ for the z's in such a way that the corresponding values $(b_{k+1}, \cdots, b_n)$ for the η's are so small that the point $(a_1, \cdots, b_n)$ lies in any preassigned neighborhood of the origin. Now take any element $P(z_1, \cdots, z_n)$ of I_n, and take any set $a_1, \cdots, a_k$, giving points inside its domain of convergence. We can find from these some value w_0 of w such that $G(w_0) = 0$, and from this construct the values $(b_{k+1}, \cdots, b_n)$. The lemma is as follows.

Lemma 1. $P(a_1, \cdots, a_k, b_{k+1}, \cdots, b_n)$ *is equal to the number obtained by taking the element of* $\Lambda_k[w]$ *corresponding to* $P(z)$*, say* $\Sigma_{i=0}^{\rho} \dfrac{c_i w^i}{D}$*, and substituting the values* $a_1, \cdots, a_k, w_0$*, in this.*

We prove the lemma by induction. We can see at once that it is true for any element independent of $(z_{k+1}, \cdots, z_n)$. Now let s be an integer in $k \leq s < n$ and for purposes of induction let us assume the lemma true for all elements independent of $(z_{s+1}, \cdots, z_n)$. Then we shall prove it true for all elements depending only on $z_1, \cdots, z_{s+1}$.

Let such an element be $P(z_1, \cdots, z_{s+1})$. We have $G_{s+1}(z_{s+1})$ an element of $\mathfrak{p}$ which is a polynomial in z_{s+1} with coefficients in I_k; it is distinguished and so we can put $P(z_1, \cdots, z_{s+1}) = H \cdot G_{s+1}(z_{s+1}) + C_1 z_{s+1}^{q-1} + \cdots + C_q$, where C_i depends only on $(z_1, \cdots, z_s)$.

When we put in the values $a_1, \cdots, a_k, b_{k+1}, \cdots, b_{s+1}$, G_{s+1} is equal to zero from the method of constructing the value b_{s+1}. So

$$P(a_1, \cdots, b_{s+1}) = C_1(a_1, \cdots, b_s) b_{s+1}^{q-1} + \cdots + C_q(a_1, \cdots, b_s)$$

But from the hypothesis of the induction $C_i(a_1, \cdots, b_s)$ is equal to the value of the corresponding element of $\Lambda_k[w]$, and b_{s+1} is precisely the value of η_{s+1}, the element corresponding to z_{s+1}; further the element corresponding to G_{s+1} is the zero element, since G_{s+1} is in $\mathfrak{p}$, so that $P(a_1, \cdots, b_{s+1})$ is equal to the value of the corresponding element of $\Lambda_k[w]$ evaluated for $a_1, \cdots, a_k, w_0$.

From this lemma it immediately follows that any set of values $(a_1, \cdots, a_k, b_{k+1}, \cdots, b_n)$, where the b's are calculated from some values of w, gives a point of the manifold of $\mathfrak{p}$.

In order to show that any point $(\bar{a}_1, \cdots, \bar{a}_k, \bar{b}_{k+1}, \cdots, \bar{b}_n)$ of the manifold for which $D(\bar{a}_1, \cdots, \bar{a}_k) \neq 0$, is a point of this kind, we consider the equation for w:

$$w = d_{k+1}(z_1, \cdots, z_k)\eta_{k+1} + \cdots + d_n(z_1, \cdots, z_k)\eta_n$$

Construct now $W(z_1, \cdots, z_n) = d_{k+1}(z_1, \cdots, z_k)z_{k+1} + \cdots + d_n(z_1, \cdots z_k)z_n$. We had that $G(w) = 0$ in $\Lambda_k[w]$, and consequently $G(W)$ is an element of $\mathfrak{p}$. Putting in the values of $(\bar{a}_1, \cdots, \bar{b}_n)$ since this is a point of the manifold, $G(W(\bar{a}_1, \cdots, \bar{b}_n)) = 0$. Let $W(\bar{a}_1, \cdots, \bar{b}_n) = w_0$, then $G(w_0) = 0$. But the coefficients of $G(w_0)$ depend only on $\bar{a}_1, \cdots, \bar{a}_k$, so that w_0 is a value of w for this set of a's.

Consider now $\eta_i = h_i(w)/D$ in $\Lambda_k[w]$; this implies that $[D(z_1, \cdots, z_k)z_i - h_i(w)]$ is an element of $\mathfrak{p}$. Therefore

$$D(\bar{a}_1, \cdots, \bar{a}_k)\bar{b}_i - h_i(w_0) = 0, \qquad \text{or} \qquad b_i = \frac{h_i(w_0)}{D(\bar{a}_1, \cdots, \bar{a}_k)}$$

This shows that the $\bar{b}$'s are derived in the normal way for some value of w dependent on the $\bar{a}$'s. In short, any zero of the manifold for which $D(a_1, \cdots, a_k) \neq 0$ is of the form

$$b_{k+1} = \frac{h_{k+1}(w_0)}{D}$$

$$\cdots$$

$$b_n = \frac{h_n(w_0)}{D}$$

where w_0 is some value of $G(w) = 0$; and any set of values of this form is a zero of the manifold.

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